

FORMALITY OF THE LITTLE N -DISKS OPERAD

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ABSTRACT. We develop the details of Kontsevich’s proof of the formality of little N -disks operad over the field of real numbers. Formality holds in the category of operads of chain complexes and also in some sense in the category of commutative differential graded algebras, which is the category encoding “real” homotopy theory. We also prove a relative version of the formality for the inclusion of the little m -disks operad in the little N -disks operad for $N \geq 2m + 1$.

CONTENTS

1. Introduction	2
1.1. Acknowledgments	8
2. Notation. Linear orders.	8
2.1. Linear orders	8
3. CDGA models for operads	8
4. Real homotopy theory of semi-algebraic sets	12
5. The Fulton-MacPherson operad	14
5.1. Compactification of configuration spaces in \mathbb{R}^N	14
5.2. The operadic structure	16
5.3. The canonical projections	17
5.4. Orientation of $C[A]$	18
5.5. Canonical projections and the operadic structure	19
5.6. Decomposition of the boundary of $C[V]$	23
6. The CDGAs of admissible diagrams	25
6.1. Diagrams	25
6.2. The module $\hat{\mathcal{D}}(A)$ of diagrams	27
6.3. Product of two diagrams	28
6.4. A differential on the space of diagrams	29
6.5. Admissible diagrams	31
7. Cooperad structures on the spaces of (admissible) diagrams	33
7.1. Construction of the structure maps $\hat{\Psi}$ and Ψ	33
7.2. $\hat{\Psi}$ and Ψ are maps of algebras	35
7.3. Ψ is a chain map	36
7.4. Associativity of the structure maps	40

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7.5.	Action of the symmetric group on the space of diagrams	40
7.6.	The cooperad structure on $\{\widehat{\mathcal{D}}(n)\}_{n \geq 0}$ and on $\{\mathcal{D}(n)\}_{n \geq 0}$	41
8.	The Kontsevich integral	41
8.1.	Construction of the Kontsevich integral $\widehat{\mathbf{I}}$	42
8.2.	$\widehat{\mathbf{I}}$ is a morphism of algebras	43
8.3.	Vanishing of $\widehat{\mathbf{I}}$ on non-admissible diagrams	44
8.4.	$\widehat{\mathbf{I}}$ and \mathbf{I} are chain maps	47
8.5.	$\widehat{\mathbf{I}}$ and \mathbf{I} are almost morphisms of cooperads	52
9.	Equivalence of the cooperads $\mathcal{D}(\bullet)$ and $H^*(C[\bullet])$	55
10.	Proof of the formality theorems	59
Appendix A.	Proof of the local triviality of the canonical projections	62
References		71

1. INTRODUCTION

In this paper we give a detailed proof of Kontsevich's theorem on the formality of the little N -disks operad [14, Theorem 2]. We also establish a relative version of formality for the inclusion morphism between the operads of little disks in different dimensions.

Our motivation for proving these results comes from applications to the study of the rational homology of the space $\text{Emb}(M, \mathbb{R}^N)$ of smooth embeddings of a compact manifold M into \mathbb{R}^N . Goodwillie-Weiss embedding calculus [10] approximates this embedding space by certain homotopical constructions based on a category of open subsets of M diffeomorphic to finitely many open balls with inclusions as morphisms. This category is closely related to the little $\dim(M)$ -disks operad. On the other hand, formality theorems can often lead to collapse results for certain spectral sequences. Combining embedding calculus with formality, the authors, along with Greg Arone, were thus able to prove in [3] the collapse of a spectral sequence computing $H_*(\overline{\text{Emb}}(M, \mathbb{R}^N); \mathbb{Q})$, where $\overline{\text{Emb}}(M, \mathbb{R}^N)$ is a slight variation of $\text{Emb}(M, \mathbb{R}^N)$. A special case of this approach also led the authors, jointly with Victor Turchin, to the proof in [16] of the collapse of the Vassiliev spectral sequence computing the rational homology of the space of long knots in \mathbb{R}^N for $N \geq 4$.

Let us explain the formality results that we prove here. Fix an integer $N \geq 1$. Recall the classical little N -disks operad $\mathcal{B}_N = \{\mathcal{B}_N(k)\}_{k \geq 0}$, where $\mathcal{B}_N(k)$ is the space of configurations of k closed N -dimensional disks with disjoint interiors contained in the unit disk of \mathbb{R}^N [4]. Integer N will usually be understood and we will just denote this operad by \mathcal{B} . This operad is homotopy equivalent to many other operads, such as the little N -cubes operad, or the Fulton-MacPherson operad $C[\bullet]$ of compactified configurations of points in \mathbb{R}^N . The latter will be important in our proofs and we will say more about it in Section 5.

The functor

$$S_*(-; \mathbb{R}): \text{Top} \rightarrow \text{Ch}_{\mathbb{R}}.$$

of singular chains with coefficients in \mathbb{R} is symmetric monoidal. Therefore $S_*(\mathcal{B}; \mathbb{R})$ is an operad of chain complexes. In addition, its homology $H_*(\mathcal{B}; \mathbb{R})$ can be viewed as an operad

of chain complexes with differential 0. One of the main results that we will prove in detail is

Theorem 1.1 (Kontsevich; Tamarkin for $N = 2$). *The little N -disks operad is stably formal over the real numbers, i.e. there exists a chain of weak equivalences of operads of chain complexes*

$$S_*(\mathcal{B}_N; \mathbb{R}) \xleftarrow{\sim} \dots \xrightarrow{\sim} H_*(\mathcal{B}_N; \mathbb{R}).$$

The proof of this theorem was sketched in [14] but the authors of this paper felt that it would be useful to develop it in full detail. Our proof seems to break down for $N = 2$, but in that dimension the formality has been proved by Tamarkin [21] using a different approach. Note that $\mathcal{B}(0)$ is in this paper the one-point space, contrary to [14] where it is the empty set.

Morally speaking, singular chains with coefficients in \mathbb{R} encode the stable “real” homotopy type of spaces or topological operads. The unstable real (or, more correctly, rational) homotopy type of spaces is encoded by commutative graded differential algebras (CDGA for short), as was discovered by Sullivan using the functor A_{PL} of polynomial forms (see Section 3). One then has the important notion of a *CDGA model* for a space X , which by definition is a CDGA weakly equivalent to $A_{PL}(X)$. Any CDGA model for X contains all the information about its rational homotopy type. We can define an analogous notion of a CDGA model of a topological operad, although the definition is a little bit more intricate (see Definition 3.1). We then have the following unstable version of Theorem 1.1.

Theorem 1.2. *For $N \neq 2$, a CDGA model over \mathbb{R} of the little N -disks operad is given by its cohomology algebra.*

We do not know whether the result is true for $N = 2$. There is only one place in the proof where the hypothesis $N \neq 2$ is needed (see Remark 8.13).

As explained in Section 3, one reason for which our definition of a CDGA model for an operad is not as direct as one might wish is that $A_{PL}(\mathcal{B})$ is not a cooperad. This is because A_{PL} is not a comonoidal functor. It might be better to consider the coalgebra of singular chains $S_*(\mathcal{B}; \mathbb{R})$, which is indeed an operad of differential coalgebras. However we do not know how to prove that this operad is weakly equivalent to its homology in the category of differential coalgebras. Moreover, that category is not very suitable for doing real homotopy theory because of the lack of strict cocommutativity.

We now state a relative version of the above theorems. Let $1 \leq m \leq N$ be integers and suppose given a linear isometry

$$\epsilon: \mathbb{R}^m \rightarrow \mathbb{R}^N.$$

Define the map

$$\mathcal{B}_\epsilon[k]: \mathcal{B}_m[k] \rightarrow \mathcal{B}_N[k]$$

that sends a configuration of k m -disks to the configuration of k N -disks where the center of each N -disk is the image under ϵ of the center of the corresponding m -disk and has the same radius. This clearly defines a morphism of operads.

Definition 1.3. A morphism of topological operads

$$\alpha: \mathcal{A} \rightarrow \mathcal{A}'$$

is *stably formal* over \mathbb{R} if there exists a zig-zag of quasi-isomorphisms of operads in $\text{Ch}_{\mathbb{R}}$ connecting the singular chains $S_*(\alpha; \mathbb{R})$ to its homology $H_*(\alpha; \mathbb{R})$ as in the following diagram

$$\begin{array}{ccccccc} S_*(\mathcal{A}; \mathbb{R}) & \xleftarrow{\simeq} & \mathcal{C}_1 & \xrightarrow{\simeq} & \cdots & \xleftarrow{\simeq} & \mathcal{C}_k \xrightarrow{\simeq} H_*(\mathcal{A}; \mathbb{R}) \\ S_*(\alpha) \downarrow & & \downarrow & & & & \downarrow H_*(\alpha) \\ S_*(\mathcal{A}'; \mathbb{R}) & \xleftarrow{\simeq} & \mathcal{C}'_1 & \xrightarrow{\simeq} & \cdots & \xleftarrow{\simeq} & \mathcal{C}'_k \xrightarrow{\simeq} H_*(\mathcal{A}'; \mathbb{R}) \end{array}$$

We say that α is *formal* over \mathbb{R} if the morphism of CDGA cooperads $H^*(\alpha; \mathbb{R})$ is a CDGA model for α (see Section 3).

Theorem 1.4. Assume that $m \geq 1$, $m \neq 2$, and $N \geq 2m + 1$. Then the morphism of operads

$$\mathcal{B}_\epsilon: \mathcal{B}_m \rightarrow \mathcal{B}_N$$

is *stably formal* and *formal* over \mathbb{R} .

There is also a notion of *coformality* which is Eckman-Hilton dual to that of (unstable) formality. Roughly speaking, coformality of a space X means that its rational homotopy type is determined by its rational homotopy Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$ (instead of its rational cohomology algebra in the case of formality). In some sense, the operad of little N -disks also seems to be coformal, although there is difficulty in making this idea precise because of the lack of a base point for the operad. We refer the reader to [2] for a discussion of coformality of the little N -disks operad.

All of the above formality results are over the field of real numbers. It would be more convenient to have rational formality because localization over \mathbb{Q} is topologically meaningful, contrary to localization over \mathbb{R} . This descent of fields for stable formality of operads is always possible when one considers *reduced operads*, that is operads in which the zero-th term (corresponding to 0-ary operation) is empty, as proved in [11, Theorem 6.2.1]. In particular we can consider the operad $\tilde{\mathcal{B}}$ which is defined by $\tilde{\mathcal{B}}(0) = \emptyset$ and $\tilde{\mathcal{B}}(k) = \mathcal{B}(k)$ for $k \geq 1$. Our formality results for \mathcal{B} are clearly also true for $\tilde{\mathcal{B}}$. Moreover, since this operad is reduced, formality for $\tilde{\mathcal{B}}$ descends to \mathbb{Q} . For our applications to embedding spaces, however, it is important to take the non-reduced little disks operad, and in this case we do not know whether formality holds over \mathbb{Q} . The proof of descent in [11, Section 6] does not generalize easily to the unreduced case because of the lack of minimal models.

We end this introduction by explaining the general idea of Kontsevich's proof of formality. The main ingredient is the construction of a certain combinatorial CDGA cooperad \mathcal{D} of *admissible diagrams* and an explicit map I to the space of differential forms on $C[k] \simeq \mathcal{B}(k)$. It will be shown in an easy algebraic argument that $\mathcal{D}(k)$ is quasi-isomorphic to the cohomology of $\mathcal{B}(k)$. From this, one then deduces that I is a quasi-isomorphism, which proves the formality.

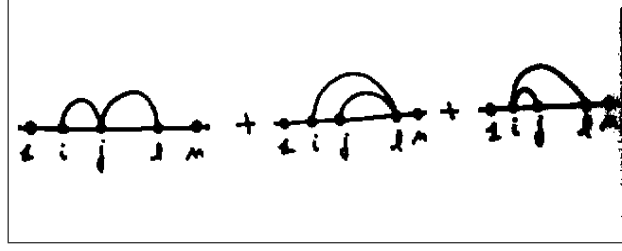


FIGURE 1. Diagrammatic description of the 3-term relation

Let us explain the ideas behind $\mathcal{D}(k)$ and I a bit further. We will work with the Fulton-MacPherson operad $\mathcal{C}[\bullet]$ which is homotopy equivalent to the little disks operad. The space $\mathcal{C}[k]$ is a compact manifold obtained by adding a boundary to the open manifold which is the space of configurations of k points in \mathbb{R}^N ,

$$F_k(\mathbb{R}^N) := \{(z_1, \dots, z_k) \in (\mathbb{R}^N)^k : z_i \neq z_j \text{ for } i \neq j\}$$

(after normalizing by modding out by translations and positive dilations). Arnold [1] computed the cohomology algebra of $F_k(\mathbb{R}^2) = F_k(\mathbb{C})$ for $N = 2$ and in fact proved that these spaces are formal in this dimension. His argument is as follows:

Consider the complex differential smooth forms

$$(1) \quad \omega_{ij} := \frac{d(z_j - z_i)}{z_j - z_i} \in \Omega_{DR}^1(F_k(\mathbb{C}); \mathbb{C})$$

which are degree 1 cocycles in the complexified De Rham complex and can easily be shown to be cohomologically independent for $1 \leq i < j \leq k$. These forms satisfy the *3-term relation*

$$(2) \quad \omega_{ij} \wedge \omega_{jl} + \omega_{jl} \wedge \omega_{li} + \omega_{li} \wedge \omega_{ij} = 0.$$

It is convenient to represent this relation by the diagram pictured in Figure 1. In this figure, the vertices on the line correspond to the labels of the points z_1, \dots, z_k of a configuration and each edge (u, v) between two vertices represent a differential form ω_{uv} .

The subalgebra of $\Omega_{DR}^*(F_k(\mathbb{C}); \mathbb{C})$ generated by the ω_{ij} is

$$\frac{\wedge(\omega_{ij} : 1 \leq i < j \leq k)}{(\omega_{ij} \wedge \omega_{jl} + \omega_{jl} \wedge \omega_{li} + \omega_{li} \wedge \omega_{ij})}.$$

This algebra has a trivial differential and it maps to the cohomology algebra. A Serre spectral sequence argument shows that this map is actually an isomorphism. In other words, the cohomology embeds in the deRham algebra of forms and $F_k(\mathbb{C})$ is formal.

This can be generalized for all N . Consider the differential forms $\omega_{ij} = \theta_{ij}^*(\text{dvol})$ where

$$\begin{aligned} \theta_{ij}: \quad F_k(\mathbb{R}^N) &\rightarrow S^{N-1} \\ (z_1, \dots, z_k) &\mapsto \frac{z_j - z_i}{\|z_j - z_i\|}, \end{aligned}$$

and dvol is the symmetric volume form on the sphere S^{N-1} that integrates to 1. For $N = 2$, these are the same as (1). It is well known by work F. Cohen in the 1970s that these forms generate the cohomology algebra of $F_k(\mathbb{R}^N)$ and that the 3-term relation holds in cohomology. However the relation is not always true at the level of forms. One only knows that, for each i, j , and l , there exists some differential form β such that

$$(3) \quad d\beta = \omega_{ij} \wedge \omega_{jl} + \omega_{jl} \wedge \omega_{li} + \omega_{li} \wedge \omega_{ij}.$$

The key idea now is to describe an algorithm which constructs *naturally* such a cobounding form β . To explain this, suppose that $k = 3$ and $(i, j, l) = (1, 2, 3)$. Consider the projection

$$\pi: F_4(\mathbb{R}^N) \rightarrow F_3(\mathbb{R}^N)$$

that forgets the fourth point of the configuration. It is a fibration with fiber $F = \mathbb{R}^N \setminus \{z_1, z_2, z_3\}$.

One can obtain β by integration along the fiber of π of some suitable form α on $F_4(\mathbb{R}^N)$. To ensure convergence of the integral, we replace the spaces in the fibration by their Fulton-MacPherson compactifications $C[4]$ and $C[3]$ so that the fiber becomes diffeomorphic to a closed disk in \mathbb{R}^N with three small open disks removed. We will denote this fiber by \bar{F} . Intuitively, each of the three inner boundary spheres of \bar{F} corresponds to points z_4 becoming infinitesimally close to z_1, z_2 , or z_3 , and the outer boundary sphere of \bar{F} corresponds to the point z_4 going to infinity.

Now consider the map

$$\theta := (\theta_{14}, \theta_{24}, \theta_{34}): C[4] \rightarrow (S^{N-1})^3$$

and the pullback form

$$\alpha = \theta^*(\text{dvol} \times \text{dvol} \times \text{dvol})$$

which is a cocycle in $\Omega_{DR}^{3N-3}(C[4])$. In other words,

$$\alpha = \omega_{14} \wedge \omega_{24} \wedge \omega_{34}.$$

Integration along the fiber is a linear map

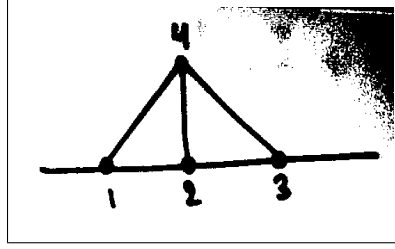
$$\begin{aligned} \pi_* = \int_{\bar{F}}: \quad \Omega_{DR}^{3N-3}(C[4]) &\rightarrow \Omega_{DR}^{2N-3}(C[3]) \\ \alpha &\mapsto \int_{\bar{F}} \alpha. \end{aligned}$$

which satisfies a fiberwise Stokes formula

$$d\left(\int_{\bar{F}} \alpha\right) = \int_{\bar{F}} d(\alpha) \pm \int_{\partial\bar{F}} \alpha.$$

The first term on the right side of the last equation vanishes because α is a cocycle. We study its second term. One of the boundary components of \bar{F} corresponds to z_4 infinitesimally close to z_1 , and we denote it by $\partial_{14}\bar{F}$ or $\{z_4 : z_1 \simeq z_4\}$. The map θ_{14} restricts to a diffeomorphism $\partial_{14}\bar{F} \cong S^{N-1}$. We have

$$\int_{z_4 \in \partial_{14}\bar{F}} \omega_{14} \wedge \omega_{24} \wedge \omega_{34} = \int_{\{z_4 : z_1 \simeq z_4\}} \omega_{14} \wedge \omega_{21} \wedge \omega_{31} = \left(\int_{S^{N-1}} \text{dvol} \right) \cdot \omega_{21} \wedge \omega_{31} = \omega_{21} \wedge \omega_{31}.$$


 FIGURE 2. The diagram Γ that cancels the 3-term relation from Figure 1.

Similarly the components corresponding to $z_2 \simeq z_4$ and $z_3 \simeq z_4$ give the two other summands of the 3-term relation. Another argument shows that the integral along the boundary corresponding to $z_4 \simeq \infty$ vanishes. Thus

$$\beta := \oint_{\bar{F}} \alpha$$

satisfies Equation (3) and is naturally defined.

This algorithm of constructing β can be encoded by a diagram Γ as pictured in Figure 2. The three edges $(1, 4)$, $(2, 4)$, and $(3, 4)$ correspond to the three components of the map θ . To such a diagram we associate the differential form

$$I(\Gamma) := \int_{\text{fiber}} \theta_{14}^*(\text{dvol}) \wedge \theta_{24}^*(\text{dvol}) \wedge \theta_{34}^*(\text{dvol})$$

where the points of the fiber are those labeled by vertices in the diagram Γ which are not on the line (z_4 in this case).

We define the coboundary of such a diagram Γ by taking the sum over all possible contraction of an edge whose one endpoint is not on the line. In particular, for Γ as in Figure 2, its coboundary is exactly the diagram of Figure 1 corresponding to the 3-term relation. In other words, I commutes with the differential. The differential vector space of all such “admissible” diagrams will be denoted by \mathcal{D} and will be endowed with the structure of a cooperad in CDGA. An algebraic computation will show that $\mathcal{D}(k)$ is quasi-isomorphic to $H^*(C[k])$, from which we will deduce that I is a quasi-isomorphism and hence that $C[\bullet]$ is formal.

There is one last technical issue. The operadic structure on $C[n]$ corresponds to the inclusions of various faces of the boundary of $C[n]$. Therefore, in order for I to be a map of cooperads, it is essential that the forms $I(\Gamma)$ are well defined on this boundary. However, the projection

$$\pi: C[k + q] \rightarrow C[k]$$

is unfortunately not a smooth submersion on the boundary $\partial C[k]$ (this can already be seen with $N = 1$, $k + q = 4$ and $k = 3$), and hence $I(\Gamma)$ need not to be a smooth form on this boundary. To fix this problem we will replace the de Rham CDGA of smooth forms Ω_{DR} by the CDGA Ω_{PA} of PA-forms as defined in [15, Appendix] and studied in great detail in [12].

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2. NOTATION. LINEAR ORDERS.

In this section we fix some notation, most of which is standard. We also review the notion of linear orders.

\mathbb{K} will be a commutative ring with unit, often \mathbb{R} .

Fix an integer $N \geq 1$ which gives the dimension of the operad disks.

For a set A we denote by $|A|$ its cardinality. We denote by $\text{Perm}(A)$ the group of permutations of A . For an integer n , we set $\underline{n} := \{1, \dots, n\}$.

When $f: X \rightarrow Y$ is a map and $A \subset X$, we denote the restriction of f to A by $f|_A$.

2.1. Linear orders.

Definition 2.1. A *linearly ordered set* is a pair (L, \leq) where L is a set and \leq is a reflexive, transitive, and antisymmetric relation on L such for any $x, y \in L$ we have $x \leq y$ or $y \leq x$. We write $x < y$ when $x \leq y$ and $x \neq y$.

Given two disjoint linearly ordered sets (L_1, \leq_1) and (L_2, \leq_2) their *ordered sum* is the linearly ordered set $L_1 \oplus L_2 := (L_1 \cup L_2, \leq)$ such that the restriction of \leq to L_i is the given order \leq_i and such that $x_1 \leq x_2$ when $x_1 \in L_1$ and $x_2 \in L_2$.

More generally if $\{L_p\}_{p \in P}$ is a family of linearly ordered sets indexed by a linearly ordered set P , its *ordered sum*

$$\bigoplus_{p \in P} L_p$$

is the disjoint union $\coprod_{p \in P} L_p$ equipped with a linear order $<$ whose restriction to each L_p is the given order on that set and such that $x < y$ when $x \in L_p$ and $y \in L_q$ with $p < q$ in P .

The *position* function on a linearly ordered finite set (L, \leq) is the unique order-preserving isomorphism

$$\text{pos}: L \rightarrow \{1, \dots, |L|\}.$$

We write $\text{pos}(x : L)$ for $\text{pos}(x)$ when we want to emphasize the underlying ordered set L .

It is clear that the ordered sum \oplus is associative but not commutative.

3. CDGA MODELS FOR OPERADS

In this section we give a precise meaning to our notion of CDGA models for a topological operad or of a morphism of topological operads. Our definition, although not difficult, is perhaps not so elegant, but it suffices for the applications we have in mind. At the end of the section we give a sketch of an alternative, more concise definition.

Recall that Sullivan [20] (see [5] or [8] for a complete development of the theory) constructed a contravariant functor of piecewise polynomial forms over a field \mathbb{K} of characteristic 0,

$$A_{PL}(-; \mathbb{K}): \text{Top} \rightarrow \text{CDGA}$$

which mimics the deRham differential algebra of smooth differential forms on a manifold. A CDGA (A, d) is a *CDGA model* (over \mathbb{K}) of a space X if the CDGAs (A, d) and $A_{PL}(X; \mathbb{K})$ are weakly equivalent, by which we mean that there exists a chain of quasi-isomorphisms of CDGAs connecting them:

$$(A, d) \xleftarrow{\simeq} \dots \xrightarrow{\simeq} A_{PL}(X; \mathbb{K}).$$

The main feature of the theory is that when X is a simply connected topological space with finite Betti numbers and $\mathbb{K} = \mathbb{Q}$, then any CDGA model of X determines the rational homotopy type of X . Moreover, many rational homotopy invariants, like the rational cohomology algebra $H^*(X; \mathbb{Q})$ or the rational homotopy Lie algebra $\pi_*(X) \otimes \mathbb{Q}$ can easily be recovered from the model (A, d) . For fields \mathbb{K} other than the rationals, we have $A_{PL}(-; \mathbb{K}) = A_{PL}(-; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{K}$, and by extension we say that the quasi-isomorphism type of $A_{PL}(X; \mathbb{K})$ determines the \mathbb{K} -homotopy type of X .

Also, if $f: X \rightarrow Y$ is a map of spaces, we say that a CDGA morphism $\phi: (B, d_B) \rightarrow (A, d_A)$ is a *CDGA model* of f if there exists a zig-zag of weak equivalences connecting ϕ and $A_{PL}(f; \mathbb{K})$, that is if there exists a diagram of CDGAs

$$\begin{array}{ccccccc} (B, d_B) & \xleftarrow{\simeq} & \bullet & \xrightarrow{\simeq} & \dots & \xleftarrow{\simeq} & \bullet & \xrightarrow{\simeq} & A_{PL}(Y; \mathbb{K}) \\ \phi \downarrow & & \downarrow & & & & \downarrow & & \downarrow A_{PL}(f; \mathbb{K}) \\ (A, d_A) & \xleftarrow{\simeq} & \bullet & \xrightarrow{\simeq} & \dots & \xleftarrow{\simeq} & \bullet & \xrightarrow{\simeq} & A_{PL}(X; \mathbb{K}) \end{array}$$

in which the horizontal arrows are quasi-isomorphisms.

We would like to define a similar notion of a CDGA model of a topological operad \mathcal{O} . A naive definition would be that such a model is a cooperad \mathcal{A} of CDGAs that is connected by weak equivalences of CDGA cooperads to $A_{PL}(\mathcal{O})$. However there is a problem with this definition because the contravariant functor A_{PL} is not comonoidal as there is no suitable natural map

$$(4) \quad A_{PL}(X \times Y) \rightarrow A_{PL}(X) \otimes A_{PL}(Y).$$

Therefore it seems that there is no naturally induced cooperad structure on $A_{PL}(\mathcal{O})$. On the other hand, A_{PL} is monoidal through the Kunnet quasi-isomorphism

$$\kappa: A_{PL}(X) \otimes A_{PL}(Y) \xrightarrow{\simeq} A_{PL}(X \times Y).$$

This morphism becomes an isomorphism in the homotopy category, and its inverse should correspond to the homotopy class of the missing map (4). We would thus like to say that $A_{PL}(\mathcal{O})$ is an cooperad “up to homotopy”. However, this sort of “up to homotopy” structure needs to be handled with more care than is necessary for our purpose, and so we will not pursue this in detail here but instead just give an indication of such a notion at the end of the section. Instead we will propose an ad hoc definition of a CDGA model of an operad.

There is a second difficulty which we will have to deal with and which comes from the proof of the formality itself. Namely, in Kontsevich's proof of the weak equivalence between the (up to homotopy) cooperad $A_{PL}(\mathcal{B})$ and its cohomology, there is use of a certain functor Ω_{PA} which is weakly equivalent to A_{PL} but is defined only after restriction to a subcategory of \mathbf{Top} , namely the category of compact semi-algebraic sets (see Section 4.) This is analogous to the fact that the deRham CDGA Ω_{DR} is weakly equivalent to $A_{PL}(-; \mathbb{R})$ after restriction to the subcategory of smooth manifolds. Consequently, our modeling functors will sometimes be defined on some subcategory $u: \mathcal{T} \hookrightarrow \mathbf{Top}$.

To finally define our notion of a CDGA model of an operad, we will need a few definitions. Two cooperads of CDGAs, \mathcal{A} and \mathcal{A}' , are *weakly equivalent* if they are connected by a chain of quasi-isomorphism of CDGA cooperads,

$$\mathcal{A} \xleftarrow{\cong} \dots \xrightarrow{\cong} \mathcal{A}'.$$

Let \mathcal{T} be a symmetric monoidal category and let

$$u: \mathcal{T} \rightarrow \mathbf{Top}$$

be a symmetric monoidal covariant functor such that

$$(5) \quad u(X) \times u(Y) \xrightarrow{\cong} u(X \times Y)$$

is an homeomorphism.

For us, a contravariant functor

$$F: \mathcal{T} \rightarrow \mathbf{CDGA}$$

is *symmetric monoidal* if it is equipped with a natural map

$$(6) \quad \kappa: F(X) \otimes F(Y) \rightarrow F(X \times Y)$$

satisfying the usual axioms and such that $F(\mathbf{1}_{\mathcal{T}}) = \mathbb{K}$.

A *natural monoidal quasi-isomorphism* between two such contravariant symmetric monoidal functors F and F' is a natural transformation

$$\theta: F \rightarrow F'$$

that induces an isomorphism in homology and that commutes with the monoidal structure maps. Two monoidal contravariant functors are *weakly equivalent* if they are connected by a chain of natural monoidal quasi-isomorphisms. Notice that if F is weakly equivalent to $A_{PL} \circ u$ then the morphism κ of (6) is a quasi-isomorphism because the corresponding one for A_{PL} is a quasi-isomorphism and because of the homeomorphism (5).

Definition 3.1. A CDGA cooperad \mathcal{A} is a *CDGA model* for a topological operad \mathcal{O} if there exist

- a CDGA cooperad \mathcal{A}' weakly equivalent to \mathcal{A} ;
- a symmetric monoidal covariant functor $u: \mathcal{T} \rightarrow \mathbf{Top}$ satisfying (5);
- an operad \mathcal{O}' in \mathcal{T} such that $u(\mathcal{O}')$ is weakly equivalent to \mathcal{O} ;
- a symmetric monoidal contravariant functor F weakly equivalent to $A_{PL} \circ u$;

- for each $n \geq 0$ a Σ_n -equivariant quasi-isomorphism

$$J_n: \mathcal{A}'(n) \rightarrow F(\mathcal{O}'(n))$$

such that, for each $k \geq 0$ and $n_1, \dots, n_k \geq 0$ with $n = n_1 + \dots + n_k$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{A}'(n) & \xrightarrow[\simeq]{J_n} & F'(\mathcal{O}'(n)) \\ \downarrow \Psi' & & \downarrow F'(\Phi') \\ & & F(\mathcal{O}'(k) \times \mathcal{O}'(n_1) \times \dots \times \mathcal{O}'(n_k)) \\ & & \uparrow \simeq \kappa \\ \mathcal{A}'(k) \otimes \mathcal{A}'(n_1) \otimes \dots \otimes \mathcal{A}'(n_k) & \xrightarrow[\simeq]{J_k \otimes J_{n_1} \otimes \dots \otimes J_{n_k}} & F(\mathcal{O}'(k)) \otimes F(\mathcal{O}'(n_1)) \otimes \dots \otimes F(\mathcal{O}'(n_k)), \end{array}$$

where Ψ' and Φ' are the (co)operad structure maps on \mathcal{A}' and \mathcal{O}' respectively, and such that the composite

$$\mathcal{A}'(1) \xrightarrow{J_1} F(\mathcal{O}'(1)) \xrightarrow{F(\eta)} F(\mathbf{1}_{\mathcal{T}}) \cong \mathbb{K}$$

is the counit of \mathcal{A}' , where η is the unit of \mathcal{O}' .

Generalizing the above in an obvious way, we say that a morphism of CDGA cooperads

$$\phi: \mathcal{B} \rightarrow \mathcal{A}$$

is a CDGA model of a morphism of topological operads

$$f: \mathcal{O} \rightarrow \mathcal{P}.$$

Definition 3.2. A topological operad is *formal* over \mathbb{K} if the induced cohomology algebra cooperad is its \mathbb{K} -CDGA model.

A morphism of topological operad is *formal* if the induced morphism in cohomology is its CDGA model.

This definition, albeit perhaps a bit ad hoc, is good enough for the applications we have in mind. A more elegant definition would have to use a precise notion of a (co)operad up to homotopy as follows.

Recall first that an operad \mathcal{O} in a symmetric monoidal category \mathcal{C} can be realized as a functor

$$\mathcal{O}: \text{Tree} \rightarrow \mathcal{C}$$

where Tree is a category whose objects are trees with a root and leaves labeled by $1, \dots, n$, and morphisms given by contractions of non-terminal edges and permutations of the labels of the vertices. The n th term $\mathcal{O}(n)$ of the operad is $\mathcal{O}(\langle n \rangle)$, where $\langle n \rangle$ is the tree whose only vertices are the root and the n leaves. Given trees S, T_1, \dots, T_k where S has k leaves and each T_i has n_i leaves, one can build a new tree $S(T_1, \dots, T_k)$ with $n_1 + \dots + n_k$ leaves

by grafting the root of each tree T_i to the corresponding leaf of S . In order for a functor O to define an operad one asks for isomorphisms

$$\alpha_{(S, T_1, \dots, T_k)}: O(S(T_1, \dots, T_k)) \xrightarrow{\cong} O(S) \otimes \otimes_{i=1}^k O(T_i)$$

satisfying obvious associativity, symmetry, and unit relations to hold.

There is a morphism in Tree given by

$$\langle k \rangle (\langle n_1 \rangle, \dots, \langle n_k \rangle) \rightarrow \langle n_1 + \dots + n_k \rangle$$

and its image under the functor O composed with the inverse of the isomorphism α gives the structure maps of the operad.

An *operad up to homotopy* is an analogous functor O except that we only ask $\alpha_{(S, T_1, \dots, T_k)}$ to be a weak equivalence instead of an isomorphism. Similarly we can define cooperads up to homotopy.

If \mathcal{O} is a topological operad, then $A_{PL}(\mathcal{O})$ becomes naturally a cooperad up to homotopy in this sense with the weak equivalences α constructed from the Kunneth quasi-isomorphism. There is also an obvious notion of morphisms of (co)operads up to homotopy and of weak equivalences. One can check that if a CDGA cooperad \mathcal{A} is a CDGA model of a topological operad \mathcal{O} in the sense of Definition 3.1, then \mathcal{A} and $A_{PL}(\mathcal{O})$ are also weakly equivalent as cooperads up to homotopy. This might thus give a better definition of an operad model.

One could also try to work with replacements by cofibrant (co)operads in order to avoid some of the problems. However there are also difficulties here. It is unclear whether there exist good cofibrant replacements in the category of CDGA cooperads. Even if one worked in the category of chain complexes instead of CDGA, the fact that we consider operads with a non-trivial term in arity 0 could be an issue for the existence of enough cofibrant models.

4. REAL HOMOTOPY THEORY OF SEMI-ALGEBRAIC SETS

In this section we give a brief review of Kontsevich and Soibelman's theory of semi-algebraic differential forms [15, §8], which we have developed in full detail in [12]. The lazy reader can safely think of Ω_{PA} in this paper as a complete analogous to Ω_{DR} , and to Ω_{\min} as some sub CDGA of it.

A *semi-algebraic set* is a subset of \mathbb{R}^p that is obtained by finite unions, finite intersections, and complements of subsets defined by polynomial equations and inequalities. A *semi-algebraic map* is a continuous map between semi-algebraic sets whose graph is a semi-algebraic set.

We will consider the categories SemiAlg (and CompactSemiAlg) of (compact) semi-algebraic sets. Endowed with the cartesian product, this category becomes symmetric monoidal and the obvious forgetful functor

$$u: \text{SemiAlg} \rightarrow \text{Top}$$

is strongly symmetric monoidal because of the natural homeomorphism

$$u(X) \times u(Y) \xrightarrow{\cong} u(X \times Y).$$

We have for a semi-algebraic set X a functorial chain complex of semi-algebraic currents $C_*(X)$ [12, Definition 3.1] whose elements in degree k are $g_*([M]) \in C_k(X)$ where $g: M \rightarrow X$ is a semi algebraic map from an oriented compact semi-algebraic manifold M .

We also have a contravariant functor of *minimal forms* [12, Section 5.2]

$$\Omega_{\min}: \text{SemiAlg} \rightarrow \text{CDGA}.$$

A minimal form of degree k on X is represented by a linear combination of

$$\mu = f_0 \cdot df_1 \wedge \cdots \wedge df_k$$

where

$$f_0, f_1, \dots, f_k: X \rightarrow \mathbb{R}$$

are semi-algebraic maps. Even though the f_i 's are maybe not everywhere smooth, for a compact semi-algebraic oriented manifold M of dimension k and a semi-algebraic map $g: M \rightarrow X$, we can always evaluate the form μ on $g_*([M])$ by

$$\langle \mu, g_*[M] \rangle := \int_M g^*(f_0 \cdot df_1 \wedge \cdots \wedge df_k).$$

It turns out that the integral on the right is always convergent because of the semi-algebraic property and compactness (see [12] for details).

The CDGA of minimal forms embeds in that of *PA forms* ([12, Section 5.4])

$$\Omega_{PA}: \text{SemiAlg} \rightarrow \text{CDGA}.$$

The important feature here is the following

Theorem 4.1 ([12, Proposition 7.1]). *When restricted to the category of compact semi-algebraic sets, the contravariant symmetric monoidal functors Ω_{PA} and $A_{PL}(u(-); \mathbb{R})$ are weakly equivalent.*

Another important feature of minimal and PA forms is that classical integration along the fiber for smooth forms can be extended in the semi-algebraic framework. To explain this, we have from [12, Section 8] an notion of *semi-algebraic bundle*, or *SA bundle* for short, which is a straightforward generalization of the usual definition of a bundle. An SA bundle

$$\pi: E \rightarrow B$$

is *oriented* if its fibers are compact oriented semi-algebraic manifolds, with orientation which is locally constant in an obvious sense.

For an oriented SA bundle with k -dimensional fiber, we have a linear map of degree $-k$ [12, Definition 8.3]

$$\pi_*: \Omega_{\min}(E)^{*+k} \rightarrow \Omega_{PA}^*(B)$$

which correponds to integration along the fiber. Properties of this map that we will need here are collected in [12, Section 8.2].

5. THE FULTON-MACPHERSON OPERAD

We review in this section the Fulton-MacPherson operad $C[\bullet] = \{C[n]\}_{n \geq 0}$, which is weakly equivalent to the little N -disks operad. Each $C[n]$ is a suitable compactification of the normalized configuration space $C(n)$ of n points in \mathbb{R}^N . It is a compact semi-algebraic manifold and the operad structure corresponds essentially to inclusions of various faces of the boundary $\partial C[n]$. We also develop some properties of the canonical projections $\pi: C[n] \rightarrow C[k]$ which consist of forgetting some points of the configuration, for $k \leq n$. In particular we will study the interaction of these canonical projections with the operadic structure.

5.1. Compactification of configuration spaces in \mathbb{R}^N . We define now the Fulton-MacPherson compactification $C[n]$ of the configuration space $C(n)$ of n points in \mathbb{R}^N . This compactification has been defined in [9] (or at least some variation of it) and alternatively by Kontsevich in [14]. We follow Kontsevich's approach which was developed by Sinha in [18].

Let A be a finite set of cardinality n . Consider the space

$$\text{Inj}(A, \mathbb{R}^N) := \{x: A \hookrightarrow \mathbb{R}^N\}$$

of all injective maps from A to \mathbb{R}^N . An element of $x \in \text{Inj}(A, \mathbb{R}^N)$ is an (ordered) configuration $(x(a))_{a \in A}$ of n distinct points in \mathbb{R}^N . This space is of course topologized as a subspace of the product $(\mathbb{R}^N)^A = \prod_{a \in A} \mathbb{R}^N$.

$\text{Inj}(A, \mathbb{R}^N)$ is a smooth open manifold of dimension $N \cdot |A|$. The semi-direct product $\mathbb{R}^N \rtimes \mathbb{R}_0^+$ acts by translation and positive dilation on \mathbb{R}^N , and hence diagonally on $\text{Inj}(A, \mathbb{R}^N)$. We denote its orbit space by

$$(7) \quad C(A) := \text{Inj}(A, \mathbb{R}^N) / (\mathbb{R}^N \rtimes \mathbb{R}_0^+).$$

When $|A| \geq 2$ the action is free and smooth and

$$\dim C(A) = N \cdot |A| - N - 1.$$

If $|A| \leq 1$ then $C(A)$ is a one point space because the action is transitive.

Note that, when $|A| \geq 2$, $C(A)$ is homeomorphic to the space

$$(8) \quad \{x: A \hookrightarrow \mathbb{R}^n \text{ such that } \sum_{a \in A} x(a) = 0 \text{ and } \sum_{a \in A} \|x(a)\| = 1\}$$

and we will from now on identify this space with $C(A)$.

Given two distinct elements $a, b \in A$ consider the map

$$(9) \quad \begin{aligned} \theta_{a,b}: C(A) &\rightarrow S^{N-1} \\ x &\mapsto \frac{x(b) - x(a)}{\|x(b) - x(a)\|} \end{aligned}$$

which gives the direction between two points in the configuration.

For three distinct elements $a, b, c \in A$ define

$$(10) \quad \begin{aligned} \delta_{a,b,c}: C(A) &\rightarrow [0, +\infty] \\ x &\mapsto \frac{\|x(a) - x(b)\|}{\|x(a) - x(c)\|} \end{aligned}$$

which gives the relative distance of 3 components of the configuration.

Set

$$\begin{aligned} A^{\{2\}} &:= \{(a, b) \in A \times A : a \neq b\} \\ A^{\{3\}} &:= \{(a, b, c) \in A \times A \times A : a \neq b \neq c \neq a\} \end{aligned}$$

and consider the map

$$\begin{aligned} \iota: C(A) &\rightarrow (S^{N-1})^{A^{\{2\}}} \times [0, +\infty]^{A^{\{3\}}} \\ x &\mapsto ((\theta_{a,b}(x))_{(a,b) \in A^{\{2\}}}, ((\delta_{a,b,c}(x))_{(a,b,c) \in A^{\{3\}}}) \end{aligned}$$

This map is a homeomorphism on its image ([18, Lemma 3.18]) and we will identify $C(A)$ with $\iota(C(A))$. The Fulton-MacPherson compactification of $C(A)$ is the topological closure of that image,

$$C[A] := \overline{\iota(C(A))}.$$

Intuitively one should think of $x \in C[A]$ as a “virtual” configuration where some points are allowed to come infinitesimally close to each other in such a way that the direction between any two points and the relative distance between 3 points is always well-defined and given by the maps $\theta_{a,b}$ and $\delta_{a,b,c}$, which obviously extend to $C[A]$.

The following notation will be useful: For a, b, c distinct in A and $x \in C[A]$, when $\delta_{a,b,c}(x) = 0$ we write

$$(11) \quad x(a) \simeq x(b) \text{ rel } x(c).$$

This happens exactly when the points $x(a)$ and $x(b)$ are infinitesimally closer to each other than to $x(c)$.

The space $C(A) \subset (\mathbb{R}^N)^A$ and the map ι are clearly semi-algebraic, therefore so is the closure $C[A]$. Moreover, by [18], $C[A]$ is a compact manifold whose interior is

$$C[A] \setminus \partial C[A] = C(A).$$

Since $C(A)$ is a semi-algebraic manifold we get that $C[A]$ is a compact semi-algebraic manifold.

In conclusion

Proposition 5.1. *For a finite set A , $C[A]$ is a compact semi-algebraic manifold with interior $C(A)$ and*

$$\dim(C[A]) = \begin{cases} 0 & \text{if } |A| \leq 1; \\ N \cdot |A| - N - 1 & \text{if } |A| \geq 2. \end{cases}$$

We also have the following important characterization of the boundary

Proposition 5.2. *For $x \in C[A]$ we have an equivalence*

$$x \in \partial C[A] \iff (\exists a, b, c \in A \text{ distinct} : x(a) \simeq x(b) \text{ rel } x(c)).$$

For $|A| \leq 1$, $C[A]$ is a one point space and, for $|A| = 2$, it is homeomorphic to the sphere S^{N-1} .

5.2. The operadic structure. The following terminology will be useful in the description of operadic structures.

Definition 5.3.

- A *partition* of a set A is a surjective map $\nu: A \rightarrow P$.
- The preimages $\nu^{-1}(p)$, for $p \in P$, are the *elements* of the partition.
- A *weak partition* is a map $\nu: A \rightarrow P$ (not necessarily surjective).
- The (weak) partition ν is *ordered* if its codomain P is equipped with a linear order.

We will use often the following setting

Setting 5.4. *Fix an ordered weak partition $\nu: A \rightarrow P$, with A and P finite. Assume that $0 \notin P$ and set*

$$(12) \quad P^* := \{0\} \odot P$$

where \odot is the ordered sum defined at Section 2.1. Set $A_p := \nu^{-1}(p)$, for $p \in P$, and $A_0 := P$.

So we have

$$C[P] \times \prod_{p \in P} C[A_p] = \prod_{p \in P^*} C[A_p]$$

where the products are taken in the linear order of their indexing sets.

There is a map

$$(13) \quad \Phi_\nu: \prod_{p \in P^*} C[A_p] \rightarrow C[A]$$

where intuitively the virtual configuration $x = \Phi((x_p)_{p \in P^*})$ is obtained by replacing the p -th component of the configuration $x_0 \in C[P]$ by the configuration $x_p \in C[A_p]$ made infinitesimal, for $p \in P$ (see [19, Figure 4.6].) More precisely this $x \in C[A]$ is characterized by

$$\theta_{a,b}(x) = \begin{cases} \theta_{a,b}(x_p) & \text{if } a, b \in A_p \text{ for some } p \in P; \\ \theta_{\nu(a), \nu(b)}(x_0) & \text{if } \nu(a) \neq \nu(b), \end{cases}$$

and

$$\delta_{a,b,c}(x) = \begin{cases} \delta_{a,b,c}(x_p) & \text{if } a, b, c \in A_p \text{ for some } p \in P; \\ \delta_{\nu(a), \nu(b), \nu(c)}(x_0) & \text{if } \nu(a), \nu(b), \text{ and } \nu(c) \text{ are all distinct;} \\ 0 & \text{if } \nu(a) = \nu(b) \neq \nu(c); \\ 1 & \text{if } \nu(a) \neq \nu(b) = \nu(c); \\ +\infty & \text{if } \nu(a) = \nu(c) \neq \nu(b). \end{cases}$$

The operad structure is now easy to define as follows

- (1) When k and n_1, \dots, n_k are non negative integers and $n = n_1 + \dots + n_k$, we have an obvious decomposition of \underline{n} into k components,

$$\underline{n} \cong \underline{n_1} \otimes \dots \otimes \underline{n_k}$$

which defines an ordered weak partition $\nu: \underline{n} \rightarrow \underline{k}$. We have an associated structure map

$$\Phi_\nu: C[k] \times C[n_1] \times \dots \times C[n_k] \rightarrow C[n].$$

- (2) There is an obvious action of the group $\text{Perm}(A)$ of permutations of the set A on $C[A]$, and in particular of Σ_n on $C[n]$.
 (3) We define the *unit* of $C[1]$ as its unique point.

Proposition 5.5. (1)-(3) above endows $\{C[n]\}_{n \geq 0}$ with the structure of an operad of semi-algebraic sets.

It is easy to see that this operad is weakly equivalent to the operad of little cubes (using for example [19, Theorem 4.9].)

5.3. The canonical projections. Let V be a finite set and let $A \subset V$ be a subset. There is an obvious map

$$(14) \quad \pi: C[V] \rightarrow C[A]$$

which consists of forgetting from the virtual configuration $y \in C[V]$ all the components corresponding to $v \in V \setminus A$. This map can also be seen as the structure map associated to the weak partition $\nu: A \hookrightarrow V$. More precisely, notice that, for $v \in V$, $\nu^{-1}(v)$ is either empty or a singleton $\{v\}$. Since $C[\emptyset]$ and $C[\{v\}]$ are both one-point spaces, the projection gives an homeomorphism

$$\text{proj}: C[V] \times \prod_{v \in V} C[\nu^{-1}(v)] \xrightarrow{\cong} C[V]$$

which we use to identify these two spaces. Then

$$C[V] = C[V] \times \prod_{v \in V} C[\nu^{-1}(v)] \xrightarrow{\Phi_\nu} C[A]$$

is the map π , which is semi-algebraic.

Definition 5.6. The map π of (14) is called the *canonical projection (associated to the inclusion $A \subset V$)*.

Theorem 5.7. Let A be a finite set and let I be a linearly ordered finite set disjoint from A . The canonical projection

$$\pi: C[A \cup I] \rightarrow C[A]$$

is an oriented semi-algebraic bundle with fiber of dimension

$$\dim(\text{fiber}(\pi)) \begin{cases} = N \cdot |I| & \text{if } |A| \geq 2 \text{ or } I = \emptyset; \\ < N \cdot |I| & \text{otherwise.} \end{cases}$$

Moreover, if $|A| \geq 2$ then:

- When N is odd the orientation of the fiber of π depends on the linear order of I . A transposition of that linear order reverses the orientation.
- When N is even the orientation of the fiber is independent of the linear order on I .

The proof of this theorem is not very difficult but it is long. Since techniques used in the proof are not used anywhere else in the paper we decided to delay it until Section A of the appendix.

5.4. Orientation of $C[A]$. In this section we fix an orientation on the manifold $C[A]$ which will be important since we will integrate over that manifold. This orientation will be canonical when N is even and will depend on a linear order on A when N is odd. We will also fix an orientation on the sphere S^{N-1}

We review first a few basic facts about orientation:

- A codimension 0 submanifold of an oriented manifold inherits that orientation;
- An orientation on a connected manifold which contains a smooth codimension 0 submanifold is determined by an equivalence class of a maximal degree differential form;
- \mathbb{R}^N is equipped with a standard orientation associated to the form $dt_1 \wedge \cdots \wedge dt_N$;
- The product of two oriented manifolds $M_1 \times M_2$ is oriented by the product of their orientation forms, $\mu_1 \times \mu_2$. Exchanging the factors preserves or reverses orientation according to the sign $(-1)^{\dim(M_1) \cdot \dim(M_2)}$;
- The boundary of an oriented manifold is oriented so that Stokes formula holds without a sign:

$$\int_{\partial M} \omega = \int_M d\omega.$$

Suppose given a linear order on A . We have then a natural orientation on the codimension 0 submanifold

$$\text{Inj}(A, \mathbb{R}^N) \subset \prod_{a \in A} \mathbb{R}^N.$$

Set

$$\text{Inj}_0(A, \mathbb{R}^N) := \{x \in \text{Inj}(A, \mathbb{R}^N) : \sum_{a \in A} x(a) = 0\}.$$

We have a homeomorphism

$$\begin{aligned} \text{Inj}_0(A, \mathbb{R}^N) \times \mathbb{R}^N &\rightarrow \text{Inj}(A, \mathbb{R}^N) \\ (x, u) &\mapsto x + u \end{aligned}$$

where $(x + u)(a) := x(a) + u$, for $a \in A$. There is a unique orientation on $\text{Inj}_0(A, \mathbb{R}^N)$ for which the above homeomorphism is of degree 1. Consider the codimension 0 submanifold

$$\text{Inj}_0^{\leq 1}(A, \mathbb{R}^N) := \{x \in \text{Inj}_0(A, \mathbb{R}^N) : \sum_{a \in A} \|x(a)\| \leq 1\}$$

with the induced orientation.

We have by (8)

$$C(A) = \partial \text{Inj}_0^{\leq 1}(A, \mathbb{R}^N)$$

and this induces our preferred orientation on $C(A)$, and hence on $C[A]$.

Consider a permutation $\sigma \in \text{Perm}(A)$ of the set A . It induces an obvious automorphism $C[\sigma]$ of the space $C[A]$. We have then:

Proposition 5.8. $C[\sigma]$ is orientation preserving or reversing according to the sign

$$(\text{sign}(\sigma))^N$$

where $\text{sign}(\sigma) = \pm 1$ is the signature of the permutation σ .

We orient the sphere S^{N-1} so that the map

$$\theta_{a,b}: C[\{a, b\}] \rightarrow S^{N-1}$$

is orientation preserving when $\{a, b\}$ is ordered by $a < b$.

5.5. Canonical projections and the operadic structure. Fix the setting 5.4. Let V be a finite set containing A and disjoint from P .

Consider the following pullback

$$(15) \quad \begin{array}{ccc} G & \xrightarrow{\Phi'} & C[V] \\ \pi' \downarrow & \text{pullback} & \downarrow \pi \\ \prod_{p \in P^*} C[A_p] & \xrightarrow{\Phi_\nu} & C[A], \end{array}$$

where π is the canonical projection and Φ_ν is the structure map (13).

In this section we will show that the map Φ' decomposes essentially as the union of various operad structure maps. AN element $g \in G$, will belong to one or another piece of the decomposition, depending on, for $y = \Phi'(g)$, how close are each component $y(i)$, for $i \in V \setminus A$, are close to the components $y(j)$, for $j \in A$. In order to make this precise we introduce first the notion of a localization.

Definition 5.9. • A *localization of V relative to the weak partition ν* is a map

$$\lambda: V \rightarrow P^*$$

such that $\lambda|_A = \nu$.

- We denote the set of these localizations by $\text{Loc}(V, \nu)$, or simply $\text{Loc}(V)$ when there is no ambiguity.
- An element $v \in V$ is *p-local* if $\lambda(v) = p$ for some $p \in P$ and it is *global* if $\lambda(v) = 0$.
- A virtual configuration $y \in C[V]$ is *λ -localized* if for each $u, v, w \in V$ we have

$$(\lambda(u) = \lambda(v) \neq 0 \text{ and } \lambda(u) \neq \lambda(w)) \implies (y(u) \simeq y(v) \text{ rel } y(w)).$$

When $V = A$ we see that ν itself is a localization of A relative to ν .

Lemma 5.10. $\text{im } \Phi_\nu = \{x \in C[A] : x \text{ is } \nu\text{-localized}\}.$

Proof. Clear. □

Let λ be a localization of V relative to ν . Set

$$G_\lambda := \{g \in G : \Phi'(g) \text{ is } \lambda\text{-localized}\}.$$

Lemma 5.11. $G = \cup_{\lambda \in \text{Loc}(V)} G_\lambda$.

Proof. It is clear that $G_\lambda \subset G$ because $\lambda|_A = \nu$ and by Lemma 5.10.

For the other direction, let $g \in G$ and set $y = \Phi'(g)$. We will construct a localization λ such that $g \in G_\lambda$. By Lemma 5.10, $\pi(y) \in C[A]$ is ν -localized. For $v \in V$ and $p \in P$ we say that g is of type (v, p) if

$$\forall a, b \in A : (\nu(a) = p \text{ and } (\nu(b) \neq p) \implies y(a) \simeq y(v) \text{ rel } y(b)).$$

It is easy to see that for each $v \in V$ there is at most one p for which g is of type (v, p) . Define

$$\lambda : V \rightarrow P^*, v \mapsto \begin{cases} p & \text{if } g \text{ is of type } (v, p) \text{ for some } p \in P; \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\lambda \in \text{Loc}(V)$ and $g \in G_\lambda$. □

Notice that the various G_λ are not necessarily pairwise disjoint. However their intersection is of smaller dimension as we will see in Lemma 5.15.

For a localization λ and $p \in P^*$ set

$$I_p = I \cap \lambda^{-1}(p) \quad \text{and} \quad V_p = A_p \cup I_p.$$

The order of I restricts to linear orders on I_p , for $p \in P^*$. Moreover we order V_0 as

$$V_0 = I_0 \oplus P.$$

The localization λ induces a weak ordered partition of V

$$\widehat{\lambda} : V \rightarrow V_0$$

defined by

$$\widehat{\lambda}(v) = \begin{cases} v & \text{if } v \in I_0; \\ \lambda(v) & \text{otherwise,} \end{cases}$$

and which induces a structure map

$$\Phi_{\widehat{\lambda}} : C[V_0] \times \left(\prod_{i \in I_0} C[\{i\}] \times \prod_{p \in P} C[V_p] \right) \rightarrow C[V].$$

Since each $C[\{i\}]$ is a one-point space, the projection

$$\text{proj} : C[V_0] \times \prod_{i \in I_0} C[\{i\}] \times \prod_{p \in P} C[V_p] \rightarrow \prod_{p \in P^*} C[V_p]$$

is a homeomorphism and by abuse of notation we will identify these two spaces without further notice.

We also have a map

$$\pi_\lambda := \prod_{p \in P^*} \pi_p : \prod_{p \in P^*} C[V_p] \rightarrow \prod_{p \in P^*} C[A_p]$$

where each π_p is the canonical projection. The pair $(\Phi_\lambda, \pi_\lambda)$ induces a map to the pullback

$$\Phi'_\lambda : \prod_{p \in P^*} C[V_p] \rightarrow G_\lambda.$$

Lemma 5.12. Φ'_λ is a homeomorphism onto G_λ .

Proof. It is clear that G_λ is the image of Φ'_λ . It is a homeomorphism because it is injective and the domain is compact. \square

We have a commutative diagram

$$(16) \quad \begin{array}{ccc} \prod_{p \in P^*} C[V_p] & \xrightarrow[\cong]{\Phi'_\lambda} & G_\lambda \\ & \searrow \pi_\lambda & \swarrow \pi'_\lambda := \pi'|_{G_\lambda} \\ & & \prod_{p \in P^*} C[A_p]. \end{array}$$

We will study now the dimension of the fibers of π'_λ . In order to do so we introduce the following

Definition 5.13. A localization λ of V relative to ν is *normal* if

$$\forall p \in P^* : |A_p| \leq 1 \implies I_p = \emptyset.$$

Lemma 5.14. Let λ be a localization relative to ν .

Then π'_λ is an oriented SA bundle with

$$\dim(\text{fiber}(\pi'_\lambda)) \begin{cases} = N \cdot |I| & \text{if } \lambda \text{ is normal;} \\ < N \cdot |I| & \text{otherwise.} \end{cases}$$

Proof. By the diagram (16) π'_λ can be identified with π_λ . Applying Theorem 5.7 to each component π_p of π_λ we get that it is an oriented bundle with fiber of the given dimension. \square

Lemma 5.15. If $\lambda_1 \neq \lambda_2$ in $\text{Loc}(V, \nu)$ then for each $x \in \prod_{p \in P^*} C[A_p]$

$$\dim \left(\pi'^{-1}_{\lambda_1}(x) \cap \pi'^{-1}_{\lambda_2}(x) \right) < N \cdot |I|.$$

Proof. Let $x \in \prod_{p \in P^*} C[A_p]$ and pick $v \in I$ such that $\lambda_1(v) \neq \lambda_2(v)$. Set $V_p = V \cap \lambda_1^{-1}(p)$ for $p \in P$ and $V_0 = \lambda_1^{-1}(0) \cup A_0$. For concreteness denote by 1 the minimum of P and suppose that $\lambda_1(v) = 1$. Set $P' = P \setminus \{1\}$. For $y \in G_{\lambda_1} \cap G_{\lambda_2}$ and for $a, b, c \in A$ with $\nu(a) = \nu(b) = 1$ and $\nu(c) \neq 1$,

$$(17) \quad y(a) \simeq y(v) \text{ rel } y(c) \quad \text{because } \lambda_1(v) = 1, \text{ and}$$

$$(18) \quad y(a) \simeq y(b) \text{ rel } y(v) \quad \text{because } \lambda_2(v) \neq 1$$

Consider the following diagram

$$\begin{array}{ccccc}
E & \xrightarrow{\hat{j}} & C[V_0] \times C[V_1] \times \prod_{p \in P'} C[V_p] & \xrightarrow{\Phi_{\lambda_1}} & C[V] \\
\widehat{\pi_1} \downarrow & \text{pullback} & \downarrow \pi_1 & & \\
C[A_0] \times (C[2] \times C[A_1] \times C[\{v\}]) \times \prod_{p \in P'} C[A_p] & \xrightarrow{j} & C[A_0] \times C[A_1 \cup \{v\}] \times \prod_{p \in P'} C[A_p] & & \\
\pi_2 \circ j \downarrow & & \downarrow \pi_2 & & \\
\prod_{p \in P^*} C[A_p] & \xlongequal{\quad} & \prod_{p \in P^*} C[A_p] & &
\end{array}$$

where $j = \text{id} \times \Phi_2 \times \text{id}$ with

$$\Phi_2: C[2] \times C[A_1] \times C[\{v\}] \rightarrow C[A_1 \cup \{v\}]$$

the structure map associated to the partition of $A_1 \cup \{v\}$ in two components A_1 and $\{v\}$, π_1 and π_2 are products of canonical projections, and the left upper square is a pullback. Notice that $\pi_2 \circ \pi_1 = \pi_{\lambda_1}$

The proximity relation (18) implies that

$$\pi_1(\Phi_{\lambda_1}^{-1}(G_{\lambda_2})) \subset \text{im } j$$

and hence

$$G_{\lambda_1} \cap G_{\lambda_2} \subset \text{im}(\hat{j} \circ \Phi_{\lambda_1}).$$

Therefore, using Lemma 5.14,

$$\begin{aligned}
\dim(\text{fiber}(\pi_{\lambda_1}) \cap G_{\lambda_2}) &\leq \dim(\text{fiber}(\pi_2 \circ \pi_1 \circ \hat{j})) \\
&= \dim(\text{fiber}(\pi_2 \circ j)) + \dim(\text{fiber}(\widehat{\pi_1})) \\
&= \dim(C[2]) + \dim(\text{fiber}(\pi_1)) \\
&\leq (N-1) + N \cdot (|I| - 1) \\
&< N \cdot |I|.
\end{aligned}$$

□

Lemma 5.16. *Let λ be a normal localization relative to ν .*

Then Φ'_λ induces homeomorphisms between the fibers of π_λ and the corresponding fibers of π'_λ , which preserve or reverse the orientation according to the sign

$$\rho(I, \lambda) := (-1)^{N \cdot |R_I|}$$

where

$$R_I := \{(v, w) \in I : v < w \text{ and } \lambda(v) > \lambda(w)\}.$$

Proof. The fact that it induces a homeomorphism between the fibers is a consequence of the diagram (16) and of Lemma 5.12.

When N is odd, the orientation of the fiber of

$$\pi_p: C[V_p] \rightarrow C[A_p]$$

depends on the linear order on $I \cap \lambda^{-1}(p) = V_p \setminus A_p$, and hence the orientation of the fiber of π_λ depends on the order of

$$(19) \quad \bigoplus_{p \in P^*} I_p.$$

On the other hand the fiber of π'_λ has the same orientation as the $(N \cdot |I|)$ -dimensional fiber of

$$\pi: C[V] \rightarrow C[A]$$

which depends on the linear order on I . The number of transpositions needed to reorder I as (19) is the cardinality of the set R_I . This gives the change of the orientation when N is odd.

When N is even the orientation is independent of the order of the factors, and hence Φ'_λ is always orientation preserving. \square

5.6. Decomposition of the boundary of $C[V]$. In this section we will show that the boundary of $C[V]$ decomposes as the union of the images of some operadic structure maps. Indeed Proposition 5.17 below give a partition of $\partial C[V]$ up to codimension 1 intersections and whose pieces are image of “ \circ_i ” operations. We will also describe the fiberwise boundary of a canonical projection in terms of operadic operations.

Let V be a finite set. For a non-empty subset W of V , let V/W be the quotient set of V , in which all the elements of W are identified in a single element, and suppose given a linear order on V/W . Consider the projection on the quotient

$$q: V \rightarrow V/W$$

which can be interpreted as an ordered partition of V . We have a structure map

$$\Phi_q: C[V/W] \times \prod_{\xi \in V/W} C[q^{-1}(\xi)] \rightarrow C[V].$$

Since $q^{-1}(\xi)$ is either a singleton $\{v\}$ or the subset W and since $C[\{v\}]$ is a one-point space, we can identify the domain of Φ_q with $C[V/W] \times C[W]$. This defines a map

$$(20) \quad \Phi_W: C[V/W] \times C[W] \rightarrow C[V]$$

that we will denote by Φ_W^V when we want to emphasize the set V . In terms of operads the map Φ_W correspond to a “circle- i ” operation (up to some permutation). Indeed when $V = \underline{n+k}$ and $W = \{i, \dots, i+k\} \cong \underline{k+1}$ then $V/W \cong \underline{n}$ and Φ_W is exactly

$$\circ_i: C[\underline{n}] \times C[\underline{k+1}] \rightarrow C[\underline{n+k}].$$

The image of Φ_W consists of configurations such that all the points labeled by W are infinitesimally closer to each other than to any point labeled by $V \setminus W$. This condition is tautological when $V = W$ or W is a singleton. Consider the set

$$\mathcal{W}(V) := \{W \subset V : W \neq V \text{ and } |W| \geq 2\}.$$

Proposition 5.17. (1) $\partial C[V] = \bigcup_{W \in \mathcal{W}(V)} \text{im}(\Phi_W)$;
 (2) for $W \in \mathcal{W}(V)$, $\dim(\text{im}(\Phi_W)) = N \cdot |V| - N - 2$;
 (3) for $W_1 \neq W_2$ in $\mathcal{W}(V)$, $\dim(\text{im}(\Phi_{W_1}) \cap \text{im}(\Phi_{W_2})) < N \cdot |V| - N - 2$.

Proof. (1) It is clear that for $W \in \mathcal{W}(V)$, $\text{im}(\Phi_W) \subset \partial C[V]$. We prove that the boundary is contained in the union of the images of the Φ_W . Let $y \in \partial C[V]$. There exist distinct elements $u_0, v_0, w_0 \in V$ such that

$$y(v_0) \simeq y(w_0) \text{ rel } y(u_0).$$

Set

$$W := \{w \in V : y(v_0) \simeq y(w) \text{ rel } y(u_0)\}.$$

Then $v_0, w_0 \in W$ and $u_0 \in V \setminus W$ and hence $W \in \mathcal{W}(V)$. Consider the canonical projections

$$\pi_1: C[V] \rightarrow C[(V \setminus W) \cup \{w_0\}] \cong C[V/W] \quad \text{and} \quad \pi_2: C[V] \rightarrow C[W].$$

Then $y = \Phi_W(\pi_1(y), \pi_2(y))$. This finishes to prove (1).

(2) For $W \in \mathcal{W}(V)$, the map Φ_W is injective and hence, by compactness, it is a homeomorphism on its image. Since $|W| \geq 2$ and $|V/W| \geq 2$, by Proposition 5.1 we have

$$\dim(\text{im } \Phi_W) = \dim C[V/W] + \dim C[W] = (N \cdot |W| - N - 1) + (N \cdot |V/W| - N - 1) = N \cdot |V| - N - 2.$$

(3) Let $W_1, W_2 \in \mathcal{W}(V)$ with $W_1 \neq W_2$. We consider three cases.

- Suppose that $W_1 \cap W_2 = \emptyset$. Then $\text{im}(\Phi_{W_1}) \cap \text{im}(\Phi_{W_2})$ is the image of the composite

$$C[(V/W_2)/W_1] \times C[W_1] \times C[W_2] \xrightarrow{(\Phi_{W_1}^{V/W_2}) \times \text{id}} C[V/W_2] \times C[W_2] \xrightarrow{\Phi_{W_2}^V} C[V]$$

and an analogous computation as in (2) implies that this image is of dimension $N \cdot |V| - N - 3$.

- Suppose that $W_1 \subset W_2$ (or the symmetric). Then $\text{im}(\Phi_{W_1}) \cap \text{im}(\Phi_{W_2})$ is the image of the composite

$$C[V/W_2] \times C[W_2/W_1] \times C[W_1] \xrightarrow{\text{id} \times (\Phi_{W_1}^{W_2})} C[V/W_2] \times C[W_2] \xrightarrow{\Phi_{W_2}^V} C[V]$$

and again this image is of dimension $N \cdot |V| - N - 3$.

- Suppose that $W_1 \cap W_2 \neq \emptyset$, $W_1 \not\subset W_2$, and $W_2 \not\subset W_1$. Choose $a \in W_1 \cap W_2$, $b \in W_1 \setminus W_2$, and $c \in W_2 \setminus W_1$. For $y \in \text{im}(\Phi_{W_1}) \cap \text{im}(\Phi_{W_2})$ we have simultaneously

$$y(a) \simeq y(b) \text{ rel } y(c) \quad \text{and} \quad y(a) \simeq y(c) \text{ rel } y(b)$$

which is impossible. Thus $\text{im}(\Phi_{W_1}) \cap \text{im}(\Phi_{W_2})$ is empty.

□

We turn now to a relative version of this decomposition of the boundary. Let $A \subset V$ and consider the canonical projection

$$\pi: C[V] \rightarrow C[A]$$

which is a bundle whose fibers are compact manifolds. The *fiberwise boundary* of π is the space

$$(21) \quad C^\partial[V] := \bigcup_{x \in C[A]} \partial(\pi^{-1}(x)).$$

We consider also the restriction map

$$\pi^\partial := (\pi|_{C^\partial[V]}): C^\partial[V] \rightarrow C[A].$$

Define the set

$$(22) \quad \mathcal{W}(V, A) := \{W \in \mathcal{W}(V) : A \subset W \text{ or } |W \cap A| \leq 1\}.$$

Proposition 5.18. $C^\partial[V] = \cup_{W \in \mathcal{W}(V, A)} \text{im } \Phi_W$.

Proof. Denote by $C(A)$ the interior of the compact manifold $C[A]$, that is

$$C(A) := C[A] \setminus \partial C[A].$$

Then

$$C^\partial[V] \cap \pi^{-1}(C(A)) = (\partial C[V]) \cap \pi^{-1}(C(A))$$

and

$$C^\partial[V] = \overline{C^\partial[V] \cap \pi^{-1}(C(A))}$$

where by \overline{A} we mean the topological closure of the subspace A .

For $W \in \mathcal{W}(V)$, if $A \not\subset W$ and $|W \cap A| \geq 2$ then $\pi(\text{im } \Phi_W) \subset \partial C[A]$ because $W \cap A \in \mathcal{W}(A)$ and $\pi(\text{im } \Phi_W)$ is in the image of

$$\Phi_{W \cap A}^A : C[A/(W \cap A)] \times C[W \cap A] \rightarrow C[A].$$

Therefore

$$\begin{aligned} C^\partial[V] &= \overline{\cup_{W \in \mathcal{W}(V)} \text{im}(\Phi_W) \cap \pi^{-1}(C(A))} \\ &= \overline{\cup_{W \in \mathcal{W}(V, A)} \text{im}(\Phi_W) \cap \pi^{-1}(C(A))} \\ &= \cup_{W \in \mathcal{W}(V, A)} \text{im}(\Phi_W). \end{aligned}$$

□

6. THE CDGAs OF ADMISSIBLE DIAGRAMS

In this section we introduce the important CDGA of *admissible diagrams*, $\mathcal{D}(A)$. We will prove latter that it is a model of both $\Omega_{PA}(C[A])$ and of its cohomology, and hence it will serve as an intermediate model in the proof of the formality. In Section 7 we will endow $\mathcal{D} := \{\mathcal{D}(n)\}_{n \geq 0}$ with the structure of a cooperad.

The CDGA $\mathcal{D}(A)$ could have been defined directly but we will describe it as a quotient of the larger CDGA of *diagrams*, $\widehat{\mathcal{D}}(A)$. One reason of doing so is that it will be easier to define a cooperad structure on $\widehat{\mathcal{D}} := \{\widehat{\mathcal{D}}(n)\}_{n \geq 0}$ and prove some its properties, and induce from that the cooperad structure for \mathcal{D} .

6.1. Diagrams. Roughly speaking, a *diagram* is a finite oriented graph where the vertices come in two flavours, *external* and *internal*, and each of the sets of internal vertices and of edges are linearly ordered. An example is represented by Figure 3 which is explained in Example 6.2 below. The precise definition is as follows:

Definition 6.1. A *diagram* Γ is a quintet $(A_\Gamma, I_\Gamma, E_\Gamma, s_\Gamma, t_\Gamma)$ where

- A_Γ is a finite set;
- I_Γ is a linearly ordered finite set disjoint from A_Γ ;
- E_Γ is a linearly ordered finite set; and

- $s_\Gamma, t_\Gamma: E_\Gamma \rightarrow A_\Gamma \cup I_\Gamma$ are functions.

We say that Γ is *diagram on* A_Γ .

Two diagrams Γ and Γ' are *isomorphic* if $A_\Gamma = A_{\Gamma'}$ and there exist order-preserving bijections $\phi_V: V_\Gamma \rightarrow V_{\Gamma'}$ and $\phi_E: E_\Gamma \rightarrow E_{\Gamma'}$ such that $\phi_V \circ s_\Gamma = s_{\Gamma'} \circ \phi_E$ and $\phi_V \circ t_\Gamma = t_{\Gamma'} \circ \phi_E$.

We fix the following terminology and notation:

- the elements of A_Γ are the *external vertices*, the elements of I_Γ are the *internal vertices*, and $V_\Gamma := A_\Gamma \amalg I_\Gamma$ is the set of all vertices. We extend the order of I_Γ to a partial order on V_Γ by letting $a < i$ when $a \in A_\Gamma$ and $i \in I_\Gamma$;
- the elements of E_Γ are the *edges*;
- $s_\Gamma(e)$ is the *source* and $t_\Gamma(e)$ is the *target* of the edge e ; both are the *endpoints* of the edge; when they are distinct we say that the two endpoints of an edge are *adjacent* vertices; we also say that the edge is *oriented* from $s_\Gamma(e)$ to $t_\Gamma(e)$;
- we partition the set of edges in the following four families:
 - a *loop* is an edge whose endpoints are identical;
 - a *chord* is an edge between two distinct external vertices;
 - a *dead end* is an edge that is not a loop and such that some of its endpoints is internal and has only one adjacent vertex;
 - a *contractible edge* is an edge that is neither a chord, nor a loop, nor a dead end;
- we denote by E_Γ^{contr} the set of contractible edges of Γ ;
- the *valence* of a vertex is the number of edges for which the vertex is an endpoint, the loops counting twice;
- an edge e is *simple* if there exists no other edge with the same set of endpoints; *double edges* are distinct edges having the same set of endpoints, *i.e.* a pair $\{e_1, e_2\}$ such that $\{s_\Gamma(e_1), t_\Gamma(e_1)\} = \{s_\Gamma(e_2), t_\Gamma(e_2)\}$;
- two vertices v and w are *connected* if there exists a path of edges joining them (not caring about the orientations,) or, in other words, if there exists a sequence of edges e_1, \dots, e_k such that $v \in \{s_\Gamma(e_1), t_\Gamma(e_1)\}$, $w \in \{s_\Gamma(e_k), t_\Gamma(e_k)\}$, and $\{s_\Gamma(e_i), t_\Gamma(e_i)\} \cap \{s_\Gamma(e_{i+1}), t_\Gamma(e_{i+1})\} \neq \emptyset$ for $1 \leq i < k$;
- a diagram is a *unit* if it has neither internal vertices, nor edges. We denote a unit by $\mathbf{1}$.

We will abuse notation by denoting by the same letter Γ a diagram and its isomorphism class.

Example 6.2. We give an example of a diagram illustrated by Figure 3. By convention all the external vertices are drawn on a line which is not part of the graph. This picture represents a diagram Γ with

- the set of external vertices is $A_\Gamma = \{1, \dots, 5\}$;
- the set of internal vertices is $I_\Gamma = \{6, \dots, 15\}$ with its natural order;
- the set E_Γ consists of eighteen edges all oriented from the lowest to the highest vertex and ordered as follows (right lexicographic order):

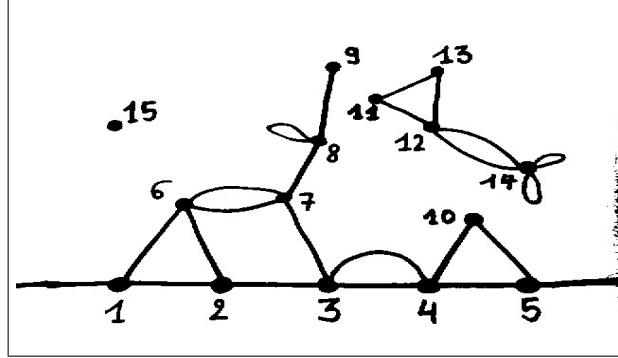


FIGURE 3. An example of a diagram (see Example 6.2.)

$$\begin{aligned}
 (3, 4) &< (1, 6) < (2, 6) < (3, 7) < (6, 7)_1 < (6, 7)_2 < \\
 &< (7, 8) < (8, 8) < (8, 9) < (4, 10) < (5, 10) < (11, 12) < \\
 &< (11, 13) < (12, 13) < (12, 14)_1 < (12, 14)_2 < (14, 14)_1 < (14, 14)_2.
 \end{aligned}$$

There are two loops at 14 and one at 8; three dead ends $(8, 9)$, $(12, 14)_1$ and $(12, 14)_2$; a chord $(3, 4)$; double contractible edges $(6, 7)_1$ and $(6, 7)_2$; and the nine others are simple contractible edges. The valence of the vertex 3 is 2, that of 8 is 4, that of 14 is 6, and that of 15 is 0.

Remark 6.3. Given two diagrams Γ_1 and Γ_2 with the same set of external vertices, we can always find a diagram Γ'_2 isomorphic to Γ_2 such that the sets I_{Γ_1} and $I_{\Gamma'_2}$, and E_{Γ_1} and $E_{\Gamma'_2}$ respectively, are disjoint. This will be used in the definition of the product of two (isomorphism classes of) diagrams in Section 6.3.

6.2. The module $\widehat{\mathcal{D}}(A)$ of diagrams. In order to define a suitable equivalence relation on the \mathbb{K} -module generated by the isomorphism classes of diagrams, we need the following:

Definition 6.4. Let Γ and Γ' be two diagrams with the same set of external vertices.

- Γ and Γ' *differ by an inversion of an edge* if, up to isomorphism, these two diagrams have the same ordered sets of internal vertices and edges, there exists an edge e such that $s_{\Gamma'}(e) = t_{\Gamma}(e)$ and $t_{\Gamma'}(e) = s_{\Gamma}(e)$, and s_{Γ} and $s_{\Gamma'}$ (respectively, t_{Γ} and $t_{\Gamma'}$) agree on all the other edges.
- Γ and Γ' *differ by a transposition in the linear order of internal vertices*, if, up to isomorphism, they have the same ordered set of edges, the same underlying set of internal vertices I , the same source and target functions, and there exists a transposition $\sigma = (a, b)$ in the group of permutations of the set I , for some pair of distinct internal vertices a and b , such that for all internal vertices $i_1, i_2 \in I$ we have that $i_1 \leq_{I_{\Gamma}} i_2$ if and only if $\sigma(i_1) \leq_{I_{\Gamma'}} \sigma(i_2)$.
- Γ and Γ' *differ by a transposition in the linear order of the edges*, if, up to isomorphism, they have the same ordered set of internal vertices, the same underlying set of edges E , the same source and target functions, and there exists a transposition $\sigma = (a, b)$ in the group of permutations of the set E , for some pair of distinct edges

a and b , such that for all edges $e_1, e_2 \in E$ we have that $e_1 \leq_{E_\Gamma} e_2$ if and only if $\sigma(e_1) \leq_{E_{\Gamma'}} \sigma(e_2)$.

Definition 6.5. The *space of diagrams on a set A* is the free \mathbb{K} -module $\widehat{\mathcal{D}}(A)$ generated by the isomorphism classes of diagrams with set of external vertices A , modulo the equivalence relation \simeq generated by the following:

- $\Gamma \simeq (-1)^N \Gamma'$ if Γ and Γ' differ by an inversion of an edge;
- $\Gamma \simeq (-1)^N \Gamma'$ if Γ and Γ' differ by a transposition in the linear order of internal vertices;
- $\Gamma \simeq (-1)^{N+1} \Gamma'$ if Γ and Γ' differ by a transposition in the linear order of edges.

When we want to emphasize the dimension N , we will denote the space of diagrams by $\widehat{\mathcal{D}}_N(A)$.

By abuse of notation we will denote by the same symbol a diagram and its equivalence class in $\widehat{\mathcal{D}}(A)$.

Definition 6.6. The *degree* of a diagram Γ is defined to be

$$\deg(\Gamma) = |E_\Gamma| \cdot (N - 1) - |I_\Gamma| \cdot N$$

where $|E_\Gamma|$ is the number of edges and $|I_\Gamma|$ is the number of internal vertices.

The degree is compatible with the equivalence relation \simeq , therefore $\widehat{\mathcal{D}}(A)$ becomes a graded \mathbb{K} -module.

6.3. Product of two diagrams. Let Γ_1 and Γ_2 be two isomorphism classes of diagrams on the same set A . We will define their product. By Remark 6.3 we can assume that the sets I_{Γ_1} and I_{Γ_2} , and E_{Γ_1} and E_{Γ_2} respectively, are disjoint. Define a diagram $\Gamma = \Gamma_1 \cdot \Gamma_2$ by

- $A_\Gamma := A$;
- $I_\Gamma := I_{\Gamma_1} \oplus I_{\Gamma_2}$;
- $E_\Gamma := E_{\Gamma_1} \oplus E_{\Gamma_2}$;
- $s_\Gamma|_{E_{\Gamma_i}} = s_{\Gamma_i}$ and $t_\Gamma|_{E_{\Gamma_i}} = t_{\Gamma_i}$,

where \oplus is the sum of linearly ordered sets defined in Section 2.1.

Example 6.7. An example of a product of two isomorphism classes of diagrams is represented in Figure 4. In each picture the edges are oriented from the lowest to the highest vertex and are ordered by the right lexicographic order, as in Example 6.2

Proposition 6.8. *The above product extends to a degree 0 linear map*

$$\widehat{\mathcal{D}}(A) \otimes \widehat{\mathcal{D}}(A) \rightarrow \widehat{\mathcal{D}}(A)$$

which endows $\widehat{\mathcal{D}}(A)$ with the structure of a graded commutative algebra.

Proof. The multiplication has been defined on generators and we extend it by bilinearity. It is clear that it is compatible with the equivalence relation on diagrams. It is clearly associative and $\deg(\Gamma_1 \cdot \Gamma_2) = \deg(\Gamma_1) + \deg(\Gamma_2)$.

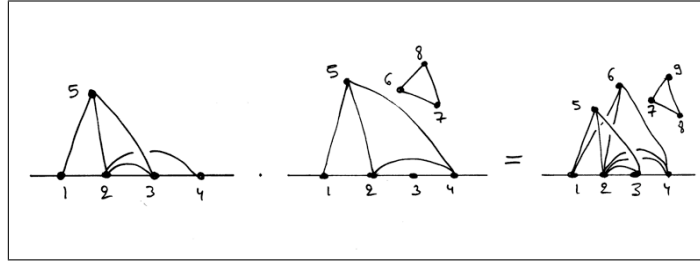


FIGURE 4. Example of a product of two diagrams

The unit diagram $\mathbf{1} = (A, \emptyset, \emptyset, \emptyset, \emptyset)$ is of degree 0 and is indeed a unit for the product. It remains to check that the multiplication is graded commutative. Let $\Gamma_i = (A, I_i, E_i, s_i, t_i)$, for $i = 1, 2$, be two diagrams. We distinguish two cases.

- Suppose that N is odd. The diagrams $\Gamma_1 \cdot \Gamma_2$ and $\Gamma_2 \cdot \Gamma_1$ differ by the order of the edges, which is irrelevant in this case, and the order of internal vertices. The number of pairs of transposed vertices is $|I_1| \cdot |I_2|$. Since N is odd, $|I_i| \equiv \deg(\Gamma_i) \pmod{2}$. Therefore $\Gamma_2 \cdot \Gamma_1 = (-1)^{\deg(\Gamma_1) \cdot \deg(\Gamma_2)} \Gamma_1 \cdot \Gamma_2$.
- Suppose that N is even. The argument is the same as for N odd after exchanging the roles of the edges and the internal vertices.

□

6.4. A differential on the space of diagrams. Recall from Definition 6.1 the notion of a contractible edge in a diagram.

Definition 6.9. Let Γ be a diagram and let e be a contractible edge of Γ . The diagram obtained from Γ by *contraction of the edge e* is the diagram $\bar{\Gamma}$ denoted by Γ/e and defined as follows:

- $A_{\bar{\Gamma}} = A_{\Gamma}$
- $I_{\bar{\Gamma}} = I_{\Gamma} \setminus \{\max(s_{\Gamma}(e), t_{\Gamma}(e))\}$
- $E_{\bar{\Gamma}} = E_{\Gamma} \setminus \{e\}$
- $s_{\bar{\Gamma}} = \pi \circ s_{\Gamma}$ and $t_{\bar{\Gamma}} = \pi \circ t_{\Gamma}$ where π is defined by:

$$\pi: V_{\Gamma} \rightarrow V_{\bar{\Gamma}} \quad , \quad v \mapsto \begin{cases} \min(s_{\Gamma}(e), t_{\Gamma}(e)) & \text{if } v = \max(s_{\Gamma}(e), t_{\Gamma}(e)) \\ v & \text{otherwise} \end{cases}$$

where the linear orders on $I_{\bar{\Gamma}}$ and $E_{\bar{\Gamma}}$ are the restrictions of those on I_{Γ} and E_{Γ} .

Notice that $\bar{\Gamma}$ is well defined because $\max(s_{\Gamma}(e), t_{\Gamma}(e))$ is internal since e is not a chord, and $\min(s_{\Gamma}(e), t_{\Gamma}(e)) \neq \max(s_{\Gamma}(e), t_{\Gamma}(e))$ since e is not a loop.

When e' is an edge distinct from the contractible edge e , we will denote by \bar{e}' the edge of $\bar{\Gamma} = \Gamma/e$ corresponding to e' in Γ through the inclusion $E_{\bar{\Gamma}} \hookrightarrow E_{\Gamma}$.

Example 6.10. An exemple of contraction of an edge is given in Figure 5 where we do not make precise the order and orientation of the edges. The edge $\overline{(8, 7)}$ is the edge $(8, 4)$ in the diagram after contraction of the edge $(4, 7)$.

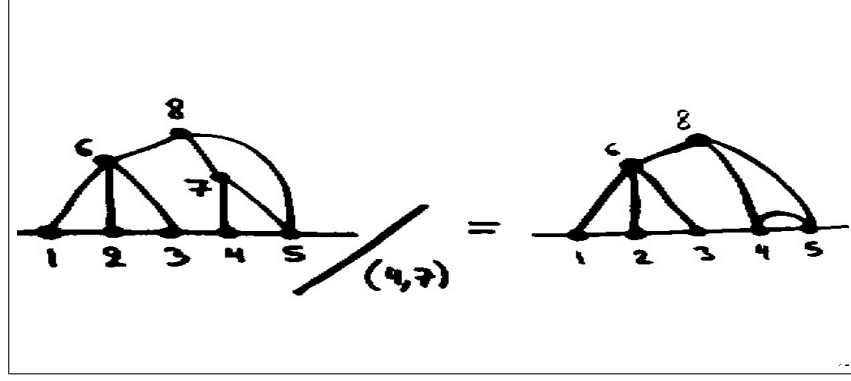


FIGURE 5. Contraction of an edge

Before defining the differential d we need to introduce a sign $\epsilon(\Gamma, e)$ associated to a diagram Γ and a contractible edge e , according to the following table

Value of $\epsilon(\Gamma, e)$	
N odd	N even
$(-1)^{\text{pos}(\max(s(e), t(e)): I_\Gamma)}$ if $s(e) < t(e)$	$(-1)^{\text{pos}(e: E_\Gamma)}$
$-(-1)^{\text{pos}(\max(s(e), t(e)): I_\Gamma)}$ if $s(e) > t(e)$	

Let Γ be a diagram on a set of external vertices A . Define its differential $d(\Gamma) \in \widehat{\mathcal{D}}(A)$ by the formula

$$(23) \quad d(\Gamma) := \sum_{e \in E_\Gamma^{\text{contr}}} \epsilon(\Gamma, e) \cdot \Gamma/e$$

where the sum runs over all contractible edges e in Γ . An example of this is the diagram Γ in Figure 2 of the Introduction for which $d(\Gamma)$ is the diagram of Figure 1 with $k = 3$, $(i, j, l) = (1, 2, 3)$, and for a suitable orientation and ordering of the edges.

Lemma 6.11. *Formula (23) induces a linear map $d: \widehat{\mathcal{D}}(A) \rightarrow \widehat{\mathcal{D}}(A)$.*

Proof. A tedious computation shows that d is compatible with the equivalence relation \simeq of Definition 6.5. \square

Lemma 6.12. *d is homogeneous of degree $+1$.*

Proof. Obvious from Definition 6.6 since, for a contractible edge e of a diagram Γ , the diagram Γ/e has one less edge and one less internal vertex than Γ . \square

Lemma 6.13. *d satisfies the Leibniz rule, $d(\Gamma \cdot \Gamma') = d(\Gamma) \cdot \Gamma' + (-1)^{\deg(\Gamma)} \Gamma \cdot d(\Gamma')$.*

Proof. Recall that $E_{\Gamma \cdot \Gamma'} = E_\Gamma \oplus E_{\Gamma'}$. It is clear that an edge is contractible in Γ or Γ' if and only if it is contractible in $\Gamma \cdot \Gamma'$. Moreover if e is a contractible edge of Γ then $(\Gamma \cdot \Gamma')/e = (\Gamma/e) \cdot \Gamma'$, and if e' is a contractible edge of Γ' then $(\Gamma \cdot \Gamma')/e' = \Gamma \cdot (\Gamma'/e')$. It remains to study the signs ϵ which appear in the differentials, which is straightforward. \square

Lemma 6.14. $d^2 = 0$.

Proof. Let Γ be a diagram and let e_1 and e_2 be distinct edges. If e_1 is contractible, denote by \bar{e}_2 the edge in Γ/e_1 corresponding to e_2 . It is easy to check that \bar{e}_2 is contractible in Γ/e_1 if and only if the following two conditions hold:

- e_1 and e_2 are contractible in Γ , and
- e_1 and e_2 have not both of their endpoints in common and if e_1 and e_2 have one endpoint in common then another endpoint of e_1 or e_2 is an internal vertex.

Since these conditions are symmetric, we deduce that \bar{e}_2 is contractible in Γ/e_1 if and only if \bar{e}_1 is contractible in Γ/e_2 , where \bar{e}_1 is the edge in Γ/e_2 corresponding to e_1 in Γ . Moreover, in that case $(\Gamma/e_1)/\bar{e}_2$ is isomorphic to $(\Gamma/e_2)/\bar{e}_1$. Therefore

$$(24) \quad d^2(\Gamma) = \sum_{e_1 < e_2} \{ \epsilon(\Gamma, e_1) \cdot \epsilon(\Gamma/e_1, \bar{e}_2) + \epsilon(\Gamma, e_2) \cdot \epsilon(\Gamma/e_2, \bar{e}_1) \} \cdot (\Gamma/e_1)/\bar{e}_2,$$

where the sum runs over each couples e_1, e_2 of distinct contractible edges of Γ such that $e_1 < e_2$ and the other condition above making \bar{e}_2 contractible in Γ/e_1 holds. It is straightforward to check that the brackets in this sum vanish. \square

Theorem 6.15. $(\widehat{\mathcal{D}}(A), d)$ is a graded commutative differential algebra.

Proof. This is a consequence of Proposition 6.8 and Lemmas 6.11-6.14. \square

6.5. Admissible diagrams.

Definition 6.16. A diagram is *admissible* if it contains no loops, no double edges, no dead ends, no internal vertices of valence ≤ 2 , and if each internal vertex is connected to some external vertex. Otherwise the diagram is *non-admissible*. We denote by $\mathcal{N}(A)$ the graded submodule of $\widehat{\mathcal{D}}(A)$ generated by the non-admissible diagrams.

Lemma 6.17. $\mathcal{N}(A)$ is a differential ideal of $\widehat{\mathcal{D}}(A)$.

Proof. It is easy to check that $\mathcal{N}(A)$ is an ideal of the algebra $\widehat{\mathcal{D}}(A)$.

We show that $\mathcal{N}(A)$ is stable by the differential d . Let Γ be a non-admissible diagram.

- If Γ contains a loop or a dead end then the same is true for each term of $d(\Gamma)$.
- If Γ contains a double edge then each term of $d(\Gamma)$ contains a double edge or a loop (when one of the double edges is contracted.)
- If Γ contains a path component with all vertices internal, then the same is true for each term of $d(\Gamma)$.
- If Γ contains an internal vertex i of valence 2 but neither double edges, nor dead ends, then for most of the terms of $d(\Gamma)$, i is still a bivalent internal vertex, except for the two terms obtained by contracting each of the two edges with endpoint i . These two terms cancel each other.
- If Γ has an internal vertex of valence 1 then it has a dead end.
- If Γ has an internal vertex of valence 0 then it has a connected component with all vertices internal.

This proves that $d(\mathcal{N}(A)) \subset \mathcal{N}(A)$. \square

Definition 6.18. The *CDGA of admissible diagrams* is the quotient $\mathcal{D}(A) := \widehat{\mathcal{D}}(A)/\mathcal{N}(A)$. We write $\mathcal{D}_N(A) = \mathcal{D}(A)$ when we want to emphasize the dimension N .

By abuse of notation we will denote by the same symbol a diagram on A , its equivalence class in $\widehat{\mathcal{D}}(A)$, and its larger equivalence class in $\mathcal{D}(A)$. The context should always remove the ambiguity.

Denote by $\mathcal{D}'(A)$ the submodule of $\widehat{\mathcal{D}}(A)$ generated by admissible diagrams. Consider

$$\mathcal{D}'(A) \xrightarrow{\iota} \widehat{\mathcal{D}}(A) \xrightarrow{\pi} \mathcal{D}(A)$$

where ι is the inclusion and π is the projection on the quotient.

Proposition 6.19. $\mathcal{D}'(A)$ is a sub-CDGA of $\widehat{\mathcal{D}}(A)$ and the composite $\pi \circ \iota$ is an isomorphism of CDGAs, $\mathcal{D}'(A) \cong \mathcal{D}(A)$.

Proof. The unit diagram is admissible. It is clear that the product of two admissible diagrams is again admissible. Also if Γ is an admissible diagram and if e is a contractible edge then Γ/e is also admissible, therefore $d(\Gamma)$ is in $\mathcal{D}'(A)$. This proves that $\mathcal{D}'(A)$ is a subCDGA of $\mathcal{D}(A)$.

It is immediate that $\pi \circ \iota$ is an isomorphism since $\widehat{\mathcal{D}}(A) = \mathcal{D}'(A) \oplus \mathcal{N}(A)$ \square

Because of this proposition we will identify $\mathcal{D}'(A)$ and $\mathcal{D}(A)$ without further notice.

Proposition 6.20. If $N \geq 3$ then $\mathcal{D}_N(A)$ is connected.

Proof. Let $\Gamma = (A, I, E, s, t)$ be an admissible diagram different than the unit. We think of an edge of Γ as the union of two half-edges, each with one endpoint which is a vertex of Γ . Since Γ is not the unit and since internal vertices are connected to some external one, there is at least one half-edge whose endpoint is an external vertex. Since each internal vertex is of valence ≥ 3 , there are at least $3 \cdot |I|$ other half-edges. Therefore $|E| \geq \frac{1}{2}(1 + 3|I|)$. We deduce that

$$\begin{aligned} \deg(\Gamma) &= |E| \cdot (N - 1) - |I| \cdot N \\ &\geq \frac{1}{2}(1 + 3|I|) \cdot (N - 1) - |I| \cdot N \\ &= \frac{N - 1}{2} + |I| \cdot \frac{N - 3}{2} > 0. \end{aligned}$$

\square

A refinement of this proves shows that, when $N \geq 4$, $\mathcal{D}_N(A)$ is $N - 2$ -connected and of finite type.

7. COOPERAD STRUCTURES ON THE SPACES OF (ADMISSIBLE) DIAGRAMS

In this section we will endow the sequence of CDGA's $\{\mathcal{D}(\underline{n})\}_{n \geq 0}$ with the structure of a cooperad. This will be obtained by endowing first $\{\widehat{\mathcal{D}}(\underline{n})\}_{n \geq 0}$ with the structure of a cooperad of graded \mathbb{K} -algebras (not differential!)

For the few next subsections, we fix the setting 5.4.

7.1. Construction of the structure maps $\hat{\Psi}$ and Ψ . In this section we build maps

$$\begin{aligned} \hat{\Psi}: \widehat{\mathcal{D}}(A) &\rightarrow \widehat{\mathcal{D}}(P) \otimes \otimes_{p \in P} \widehat{\mathcal{D}}(A_p) \quad \text{and} \\ \Psi: \mathcal{D}(A) &\rightarrow \mathcal{D}(P) \otimes \otimes_{p \in P} \mathcal{D}(A_p). \end{aligned}$$

which will serve as structure maps for the cooperadic structure. Of course the tensor product over $p \in P$ is taken in the order fixed on P . Since $A_0 = P$ we have

$$\widehat{\mathcal{D}}(P) \otimes \otimes_{p \in P} \widehat{\mathcal{D}}(A_p) = \otimes_{p \in P^*} \widehat{\mathcal{D}}(A_p).$$

Definition 7.1. Let Γ be a diagram on A and assume that $I_\Gamma \cap P = \emptyset$.

- A *localization* λ on Γ is a localization of V_Γ relative to ν as in Definition 5.9. We set $\text{Loc}(\Gamma) := \text{Loc}(V_\Gamma)$
- The *extension to the edges* of the localization λ on Γ is the map

$$\lambda_E: E_\Gamma \rightarrow P^*$$

defined by

$$\lambda_E(e) = \begin{cases} \lambda(s_\Gamma(e)) & \text{if } \lambda(s_\Gamma(e)) = \lambda(t_\Gamma(e)), \\ 0 & \text{otherwise.} \end{cases}$$

- Given a localization λ of Γ , a vertex v (respectively an edge e) is *p-local*, for $p \in P$, if $\lambda(v) = p$ (respectively $\lambda_E(e) = p$). It is *global* if $\lambda(v) = 0$ (respectively $\lambda_E(e) = 0$).

Clearly the set of localizations on Γ is in bijection with the set of maps from I_Γ to P^* , since the value of a localization λ on an external vertex a is determined by $\lambda(a) = p$ for $a \in A_p$.

Let Γ be a diagram on A and let $\lambda \in \text{Loc}(\Gamma)$. Assume that $I_\Gamma \cap P = \emptyset$. For $p \in P^*$ we define a diagram

$$(25) \quad \Gamma(\lambda, p) := (A_p, I_p, E_p, s_p, t_p)$$

with

- $I_p = I_\Gamma \cap \lambda^{-1}(p)$;
- $E_p = \lambda_E^{-1}(p)$;
- – For $p \in P$, s_p and t_p are the restrictions of s_Γ and t_Γ to E_p ;
- – For $p = 0$, $s_0 = \widehat{\lambda} \circ s_\Gamma$ and $t_0 = \widehat{\lambda} \circ t_\Gamma$ where

$$\widehat{\lambda}: V_\Gamma \rightarrow P \cup I_0$$

is defined by $\widehat{\lambda}(v) = v$ if $\lambda(v) = 0$, and $\widehat{\lambda}(v) = \lambda(v)$ otherwise.

For $p \in P$ the diagram $\Gamma(\lambda, p)$ is the diagram obtained from Γ by only keeping its p -local vertices and the edges between them. The diagram $\Gamma(\lambda, 0)$ should be think of as the diagram obtained from Γ by drawing all p -local vertices infinitesimally close to each other so that they become a single external vertex p , for each $p \in P$, and keeping all the global internal vertices and all the edges which did not became infinitesimal.

It is clear that the sets of edges (respectively of internal vertices) of Γ is the disjoint union for $p \in P^*$ of the set of edges (respectively of internal vertices) of the diagrams $\Gamma(\lambda, p)$.

Note that the equivalence class of $\Gamma(\lambda, p)$ in $\widehat{\mathcal{D}}(A_p)$, or even $\otimes_{p \in P^*} \Gamma(\lambda, p)$, is *not* an invariant of the equivalence class of Γ in $\widehat{\mathcal{D}}(A)$. To correct this we introduce a sign

$$(26) \quad \rho(\Gamma, \lambda) := (-1)^{(N-1) \cdot |R_E| - N \cdot |R_I|}$$

where

$$\begin{aligned} R_I &:= \{(v, w) \in I_\Gamma \times I_\Gamma : v < w \text{ and } \lambda(v) > \lambda(w)\} \\ R_E &:= \{(e, f) \in E_\Gamma \times E_\Gamma : e < f \text{ and } \lambda_E(e) > \lambda_E(f)\} \end{aligned}$$

Lemma 7.2. *For a diagram Γ and a localization λ on Γ the element*

$$\rho(\Gamma, \lambda) \cdot \otimes_{p \in P^*} \Gamma(\lambda, p) \in \otimes_{p \in P^*} \widehat{\mathcal{D}}(A_p)$$

depends only on the equivalence class of Γ in $\widehat{\mathcal{D}}(A)$.

Proof. Straightforward. □

For a diagram Γ on A and a localization λ of Γ we set

$$(27) \quad \Gamma(\lambda) := \rho(\Gamma, \lambda) \cdot \otimes_{p \in P^*} \Gamma(\lambda, p) \quad \text{in} \quad \otimes_{p \in P^*} \widehat{\mathcal{D}}(A_p).$$

By Lemma 7.2 we get a linear map

$$\hat{\Psi}: \widehat{\mathcal{D}}(A) \rightarrow \widehat{\mathcal{D}}(P) \otimes \otimes_{p \in P} \widehat{\mathcal{D}}(A_p)$$

defined on generators by

$$(28) \quad \hat{\Psi}(\Gamma) := \sum_{\lambda \in \text{Loc}(\Gamma)} \Gamma(\lambda).$$

Recall $\mathcal{N}(A_p) \subset \widehat{\mathcal{D}}(A_p)$ which is the ideal of non-admissible diagrams (Lemma 6.17). Set

$$\tilde{\mathcal{N}} := \sum_{p \in P^*} \otimes_{q \in P^*} \widehat{\mathcal{D}}(A_q) \otimes \mathcal{N}(A_p) \otimes \otimes_{q \in P^*} \widehat{\mathcal{D}}(A_q)$$

which is a differential ideal in $\otimes_{p \in P^*} \widehat{\mathcal{D}}(A_p)$. Since $\mathcal{D}(A_p) = \widehat{\mathcal{D}}(A_p)/\mathcal{N}(A_p)$, we have an isomorphism of CDGA

$$(29) \quad \left(\otimes_{p \in P^*} \widehat{\mathcal{D}}(A_p) \right) / \tilde{\mathcal{N}} \cong \otimes_{p \in P^*} \mathcal{D}(A_p)$$

Lemma 7.3. $\hat{\Psi}(\mathcal{N}(A)) \subset \tilde{\mathcal{N}}$.

Proof. Let Γ be a non-admissible diagram on A and let $\lambda \in \text{Loc}(\Gamma)$.

- If Γ has a loop at a vertex v then $\Gamma(\lambda, \lambda(v))$ has also a loop.
- If Γ has double edges e_1 and e_2 then so does $\Gamma(\lambda, \lambda_E(e_1))$.
- If Γ has an internal vertex v of valence ≤ 2 then the same is true for $\Gamma(\lambda, \lambda(v))$ because the valence of v can only decrease.
- If Γ has a dead end then it has an internal vertex of valence 1, or a loop, or double edges.
- If Γ has an internal p -local vertex v that is not connected to any external vertex, for some $p \in P$, then the same is true for $\Gamma(\lambda, p)$. If Γ has a connected component consisting only of internal global vertices then the same is true for $\Gamma(\lambda, 0)$.

In all cases we see that if Γ is not admissible then the same is true for $\Gamma(\lambda, p)$ for some $p \in P^*$. Therefore $\Gamma(\lambda) \in \tilde{\mathcal{N}}$ and $\hat{\Psi}(\mathcal{N}(A)) \subset \tilde{\mathcal{N}}$. \square

Proposition 7.4. $\hat{\Psi}$ induces a linear map

$$\Psi: \mathcal{D}(A) \rightarrow \mathcal{D}(P) \otimes \otimes_{p \in P} \mathcal{D}(A_p).$$

Proof. This is an immediate consequence of the isomorphism (29) and of Lemma 7.3. \square

7.2. $\hat{\Psi}$ and Ψ are maps of algebras. The aim of this section is to prove

Proposition 7.5. $\hat{\Psi}$ and Ψ are morphisms of algebras.

Proof of Proposition 7.5. We prove it first for $\hat{\Psi}$. Let Γ_1 and Γ_2 be two diagrams on A and suppose that I_{Γ_1} and I_{Γ_2} , E_{Γ_1} and E_{Γ_2} respectively, are disjoint.

Consider the function

$$\text{Loc}(\Gamma_1) \times \text{Loc}(\Gamma_2) \rightarrow \text{Loc}(\Gamma_1 \cdot \Gamma_2) \quad , \quad (\lambda_1, \lambda_2) \mapsto \lambda_1 \cdot \lambda_2$$

defined by $(\lambda_1 \cdot \lambda_2)(v) := \lambda_i(v)$ when $v \in V_{\Gamma_i}$ for $i = 1, 2$. This map is well defined because if $v \in V_{\Gamma_1} \cap V_{\Gamma_2}$ then v is external and $\lambda_1(v) = \lambda_2(v)$. Moreover it is a bijection whose inverse is given by $\lambda \mapsto (\lambda|_{V_{\Gamma_1}}, \lambda|_{V_{\Gamma_2}})$.

Since

$$\Gamma_1(\lambda_1, p) \cdot \Gamma_2(\lambda_2, p) = (\Gamma_1 \cdot \Gamma_2)(\lambda_1 \cdot \lambda_2, p)$$

it is easy to see that

$$\otimes_{p \in P^*} (\Gamma_1 \cdot \Gamma_2)(\lambda_1 \cdot \lambda_2, p) = \eta(\Gamma_1, \lambda_1, \Gamma_2, \lambda_2) \cdot (\otimes_{p \in P^*} \Gamma_1(\lambda_1, p)) \cdot (\otimes_{q \in P^*} \Gamma_2(\lambda_2, q)).$$

where

$$\eta(\Gamma_1, \lambda_1, \Gamma_2, \lambda_2) := (-1)^{\left\{ \sum_{\substack{p, q \in P^* \\ q < p}} \deg(\Gamma_1(\lambda_1, p)) \cdot \deg(\Gamma_2(\lambda_2, q)) \right\}}$$

We have

$$\begin{aligned}
\hat{\Psi}(\Gamma_1 \cdot \Gamma_2) &= \sum_{\lambda \in \text{Loc}(\Gamma_1 \cdot \Gamma_2)} \rho(\Gamma_1 \cdot \Gamma_2, \lambda) \cdot \otimes_{p \in P^*}(\Gamma_1 \cdot \Gamma_2)(\lambda, p) \\
&= \sum_{\lambda_1 \in \text{Loc}(\Gamma_1)} \sum_{\lambda_2 \in \text{Loc}(\Gamma_2)} \rho(\Gamma_1 \cdot \Gamma_2, \lambda_1 \cdot \lambda_2) \cdot \otimes_{p \in P^*}(\Gamma_1 \cdot \Gamma_2)(\lambda_1 \cdot \lambda_2, p) \\
&= (30) \sum_{\lambda_1 \in \text{Loc}(\Gamma_1)} \sum_{\lambda_2 \in \text{Loc}(\Gamma_2)} \rho(\Gamma_1 \cdot \Gamma_2, \lambda_1 \cdot \lambda_2) \cdot \eta(\Gamma_1, \lambda_1, \Gamma_2, \lambda_2) \cdot (\otimes_{p \in P^*} \Gamma_1(\lambda_1, p)) \cdot (\otimes_{q \in P^*} \Gamma_2(\lambda_2, q))
\end{aligned}$$

On the other hand

$$\begin{aligned}
\hat{\Psi}(\Gamma_1) \cdot \hat{\Psi}(\Gamma_2) &= \sum_{\lambda_1 \in \text{Loc}(\Gamma_1)} \sum_{\lambda_2 \in \text{Loc}(\Gamma_2)} \Gamma_1(\lambda_1) \cdot \Gamma_2(\lambda_2) \\
(31) \quad &= \sum_{\lambda_1 \in \text{Loc}(\Gamma_1)} \sum_{\lambda_2 \in \text{Loc}(\Gamma_2)} \rho(\Gamma_1, \lambda_1) \cdot \rho(\Gamma_2, \lambda_2) \cdot (\otimes_{p \in P^*} \Gamma_1(\lambda_1, p)) \cdot (\otimes_{q \in P^*} \Gamma_2(\lambda_2, q))
\end{aligned}$$

It remains to check that the signs of (30) and (31) agree, which is straightforward..

For Ψ the proposition is then a consequence of the definition of Ψ in Proposition 7.4 and of the fact that (29) is an isomorphism of algebras. \square

7.3. Ψ is a chain map. All this section is devoted to the proof of the following:

Proposition 7.6. *Ψ commutes with the differentials.*

Notice that it is *not* true that $\hat{\Psi}$ commutes with the differential.

Let Γ be a diagram on A and let λ be a localization on Γ .

Definition 7.7. An edge e of Γ is λ -contractible if it is contractible in $\Gamma(\lambda, \lambda_E(e))$.

Lemma 7.8. *An edge e of Γ is λ -contractible if and only if the following conditions hold:*

- (1) e is contractible in Γ , and
- (2) $\lambda(s_\Gamma(e)) = \lambda(t_\Gamma(e))$ or $\min(\lambda(s_\Gamma(e)), \lambda(t_\Gamma(e))) = 0$.

Proof. Easy. \square

Let e be a contractible edge in Γ . Let v and w be the endpoints of e with $v < w$. Thus $V_{\Gamma/e} = V_\Gamma \setminus \{w\}$. Define the function

$$\lambda/e: V_{\Gamma/e} \rightarrow P^*$$

by

$$(\lambda/e)(z) = \begin{cases} \lambda(z) & \text{if } z \neq v \text{ or } z = v \text{ is external} \\ \max(\lambda(v), \lambda(w)) & \text{if } z = v \text{ is internal} \end{cases}$$

It is clear that λ/e is a localization on Γ/e . Notice also that if e is λ -contractible then $(\lambda/e)(v) = \max(\lambda(v), \lambda(w))$.

Definition 7.9. A localization λ of Γ is *regular* if for each internal vertex v and each $p \in P$ we have the equivalence

$$\lambda(v) = p \iff v \text{ admits at least two distinct adjacent } p\text{-local vertices.}$$

Denote by $\text{RegLoc}(\Gamma)$ the set of regular localization on Γ .

Lemma 7.10. *If λ is not regular then $\Gamma(\lambda) \in \tilde{\mathcal{N}}$.*

Proof. Suppose that v is a p -local internal vertex, for some $p \in P$, and suppose that it does not have two adjacent p -local vertices. Then v is internal of valence < 2 in $\Gamma(\lambda, p)$, and hence $\Gamma(\lambda, p) \in \mathcal{N}(A_p)$.

Suppose that v is an internal vertex that is not p -local but that has two adjacent p -local vertices, for some $p \in P$. Then in $\Gamma(\lambda, 0)$, the external vertex p is connected by a double edge to either v (if $\lambda(v) = 0$) or to the external vertex q (if $\lambda(v) = q \in P \setminus \{p\}$). Thus $\Gamma(\lambda, 0) \in \mathcal{N}(P)$. \square

Lemma 7.10 implies that, in $\otimes_{p \in P^*} \mathcal{D}(A_p)$ and for Γ admissible,

$$(32) \quad \Psi(\Gamma) = \sum_{\lambda \in \text{RegLoc}(\Gamma)} \Gamma(\lambda).$$

Lemma 7.11. *Let λ_1, λ_2 be two regular localizations on Γ .*

If λ_1 and λ_2 coincide on all vertices except maybe one, then $\lambda_1 = \lambda_2$.

Proof. Let u be a vertex of Γ such that $\lambda_1(v) = \lambda_2(v)$ for $v \neq u$. If u is external then the values of $\lambda_i(u)$ are determined and hence $\lambda_1 = \lambda_2$. Suppose that u is internal. If u has two adjacent vertices that are p -local (for both λ_1 and λ_2), for some $p \in P$, then $\lambda_i(u) = p$, by regularity. Otherwise $\lambda_i(u) = 0$, again by regularity. \square

Let Γ be an admissible diagram. Consider the sets

$$\begin{aligned} \Omega &= \{(e, \lambda) : e \in E_\Gamma, \lambda \in \text{RegLoc}(\Gamma), e \text{ } \lambda\text{-contractible}, (\Gamma/e)(\lambda/e) \neq 0 \text{ in } \otimes_{p \in P^*} \mathcal{D}(A_p)\}, \\ \bar{\Omega} &= \{(e, \bar{\lambda}) : e \in E_\Gamma, e \text{ contractible}, \bar{\lambda} \in \text{Loc}(\Gamma/e), (\Gamma/e)(\bar{\lambda}) \neq 0 \text{ in } \otimes_{p \in P^*} \mathcal{D}(A_p)\}, \end{aligned}$$

and the map

$$\omega : \Omega \rightarrow \bar{\Omega}, (e, \lambda) \mapsto (e, \lambda/e).$$

Lemma 7.12. *ω is a bijection.*

Proof. We show first that ω is injective. Let e be a contractible edge and, for $i = 1, 2$, let λ_i be regular localizations of Γ such that e is λ_i -contractible and $(\Gamma/e)(\lambda_i/e) \neq 0$. Assume that $\lambda_1/e = \lambda_2/e$. We will show that $\lambda_1 = \lambda_2$.

Set $\bar{\lambda} = \lambda_1/e$ which is regular because $(\Gamma/e)(\bar{\lambda}) \neq 0$ in $\otimes_{p \in P^*} \mathcal{D}(A_p)$ and of Lemma 7.10. Let v and w be the endpoints of e with $v < w$. Thus $V_{\Gamma/e} = V_\Gamma \setminus \{w\}$. We know that λ_i agrees with $\bar{\lambda}$ on $V_\Gamma \setminus \{v, w\}$, therefore we only need to show that $\lambda_1(v) = \lambda_2(v)$ and $\lambda_1(w) = \lambda_2(w)$. Moreover, since each λ_i are regular, by Lemma 7.11 it is enough to prove only one of these two equations.

If v is external then $\lambda_1(v) = \lambda_2(v)$ is determined and hence $\lambda_1 = \lambda_2$. Suppose that v is internal. If $\bar{\lambda}(v) = 0$ then, since $\bar{\lambda}(v) = \max(\lambda_i(v), \lambda_i(w))$, we get $\lambda_i(v) = \lambda_i(w) = 0$, for $i = 1, 2$, and hence $\lambda_1 = \lambda_2$. Suppose that $\bar{\lambda}(v) = p \in P$. By the regularity of $\bar{\lambda}$ there exists two vertices x and y other than v and w that are adjacent to v in Γ/e and with $\bar{\lambda}(x) = \bar{\lambda}(y) = p$. This implies that $\lambda_i(x) = \lambda_i(y) = p$, for $i = 1, 2$. Then in Γ , either x and y are both adjacent to v (respectively to w), or x is adjacent to v and y is adjacent to w (or the symmetric.) In the first case we get by regularity that $\lambda_1(v) = \lambda_2(v) = p$ (respectively $\lambda_1(w) = \lambda_2(w) = p$), and hence $\lambda_1 = \lambda_2$ by Lemma 7.11. In the second case, since $p = \bar{\lambda}(v) = \max(\lambda_1(v), \lambda_1(w))$, we get that $\lambda_1(v) = p$ or $\lambda_1(w) = p$. Let say that $\lambda_1(v) = p$, the other case being completely analogous. Then w is adjacent to v and to either x or y , thus w is adjacent to two p -local vertices (for the localization λ_1), and hence we also have $\lambda_1(w) = p$ by regularity. The same argument shows that $\lambda_2(v) = \lambda_2(w) = p$. This achieves to prove the injectivity of ω .

We prove that ω is surjective. Let e be a contractible edge of Γ and let $\bar{\lambda}$ be a localization of Γ/e such that $(\Gamma/e)(\bar{\lambda}) \neq 0$. We will construct a regular localization λ of Γ such that e is λ -contractible and $\lambda/e = \bar{\lambda}$. Let again v and w be the endpoints of e with $v < w$. For $z \in V_\Gamma \setminus \{v, w\}$, set $\lambda(z) = \bar{\lambda}(z)$. We need to define $\lambda(v)$ and $\lambda(w)$ and to check that λ has the desired properties. We treat different cases.

- (1) Assume that v is external. Then $\lambda(v) = \bar{\lambda}(v) = p \in P$ is prescribed.
 - (a) Suppose that there exists a vertex x in Γ distinct of v and adjacent to w such that $\bar{\lambda}(x) = p$. In that case, set $\lambda(v) = \lambda(w) = p$. Then λ is regular at the vertex w because it has two p -local adjacent vertices v and x . It is easy to check that λ is also regular at the other internal vertices of Γ , using the fact that $\bar{\lambda}$ is. Moreover e is λ -contractible since it is contractible and $\lambda(v) = \lambda(w)$.
 - (b) Suppose that w is not adjacent to any p -local vertex other than v . In that case, set $\lambda(v) = p$ and $\lambda(w) = 0$. The vertex w is not adjacent in Γ to two vertices x and y such that $\bar{\lambda}(x) = \bar{\lambda}(y) = q \in P$ with $p \neq q$ because otherwise $(\Gamma/e)(\bar{\lambda}, 0)$ would contain a double edge joining p and q , and hence $\Gamma/e \in \tilde{\mathcal{N}}$, contrary to our hypothesis. This proves that λ is regular at w and the regularity at other vertices is a consequence of the regularity of $\bar{\lambda}$. Also e is λ -contractible.
- (2) Suppose that v is internal. Then w is also internal since $v < w$.
 - (a) Suppose that $\bar{\lambda}(v) = 0$. In that case set $\lambda(v) = \lambda(w) = 0$. By regularity of $\bar{\lambda}$, there do not exists two vertices $x, y \in V_\Gamma \setminus \{v, w\}$ adjacent in Γ to either v or w with $\bar{\lambda}(x) = \bar{\lambda}(y) \in P$. It is easy to see that λ is regular and e is λ -contractible.
 - (b) Suppose that $\bar{\lambda}(v) = p \in P$. By regularity of $\bar{\lambda}$ there exist two distinct vertices $x, y \in V_\Gamma \setminus \{v, w\}$ adjacent in Γ to either v or w such that $\bar{\lambda}(x) = \bar{\lambda}(y) = p$.
 - If v is not adjacent to any vertices in $(V_\Gamma \setminus \{v, w\}) \cap \bar{\lambda}^{-1}(p)$ then set $\lambda(v) = 0$ and $\lambda(w) = p$.
 - If w is not adjacent to any vertices in $(V_\Gamma \setminus \{v, w\}) \cap \bar{\lambda}^{-1}(p)$ then set $\lambda(v) = p$ and $\lambda(w) = 0$.
 - If both v and w are adjacent to some vertices in $(V_\Gamma \setminus \{v, w\}) \cap \bar{\lambda}^{-1}(p)$ then set $\lambda(v) = \lambda(w) = p$.

In each case it is easy to see that λ is regular and that e is λ -contractible.

This proves the surjectivity of ω . \square

Lemma 7.13. *If Γ is admissible, if λ is regular, and if e is a λ -contractible edge of Γ , then, for $p \in P^*$, we have in $\mathcal{D}(A_p)$:*

$$(\Gamma/e)(\lambda/e, p) = \begin{cases} \Gamma(\lambda, p)/e & \text{if } p = \lambda_E(e) \\ \Gamma(\lambda, p) & \text{otherwise} \end{cases}$$

Proof. Let v and w be the endpoints of e with $v < w$. Then $V_{\Gamma/e} = V_\Gamma \setminus \{w\}$ and $E_{\Gamma/e} = E_\Gamma \setminus \{e\}$. It is easy to see that the equations to prove are equivalent to

$$\begin{cases} \lambda/e &= \lambda|_{V_\Gamma \setminus \{w\}} \\ (\lambda/e)_E &= \lambda_E|_{E_\Gamma \setminus \{e\}} \end{cases}$$

Since e is λ -contractible, by Lemma 7.8, $\lambda(v) = \lambda(w)$ or $\min(\lambda(v), \lambda(w)) = 0$. If $\lambda(v) < \lambda(w)$ then $\lambda(v) = 0$ which implies that v is internal, and the same for w because $v < w$, in which case we can transpose the order of v and w to get an equivalent diagram (up to sign) in which the roles of v and w are exchanged. Therefore, without loss of generality we can always assume that $\lambda(v) \geq \lambda(w)$. This implies that $(\lambda/e)(v) = \lambda(v)$. Also for $z \neq v, w$ we have $(\lambda/e)(z) = \lambda(z)$. Thus $\lambda/e = \lambda|_{V_\Gamma \setminus \{w\}}$.

It remains to prove that $(\lambda/e)_E = \lambda_E|_{E_\Gamma \setminus \{e\}}$. Let $f \neq e$ be an edge of Γ . If w is not an endpoint of f then, since $\lambda/e = \lambda|_{V_\Gamma \setminus \{w\}}$, it is clear that $(\lambda/e)_E(f) = \lambda_E(f)$. Suppose that w is an endpoint of f . If $\lambda(w) = \lambda(v)$ then it is clear that $(\lambda/e)_E(f) = \lambda_E(f)$. Otherwise $\lambda(w) = 0$ and $\lambda(v) = p \in P$, and hence f is global in Γ . As Γ is admissible and $f \neq e$, the other endpoint of f is not v . Since λ is regular and since w is not r -local but is adjacent to the p -local vertex v , we get that the other endpoint of f is not p -local. This implies that f is global in Γ/e , and hence $(\lambda/e)_E(f) = \lambda_E(f) = 0$. This achieves to prove that $(\lambda/e)_E = \lambda_E|_{E_\Gamma \setminus \{e\}}$. \square

Proof of Proposition 7.6. Let Γ be an admissible diagram. For $p \in P^*$ and for a localization λ of Γ , define the sign

$$\eta(\Gamma, \lambda, p) := (-1)^{\left\{ \sum_{\substack{q \in P^* \\ q < p}} \deg(\Gamma(\lambda, q)) \right\}}.$$

We have

$$\begin{aligned} d(\Psi(\Gamma)) &\stackrel{\text{Equation (32)}}{=} d \left(\sum_{\lambda \in \text{RegLoc}(\Gamma)} \Gamma(\lambda) \right) \\ &= \sum_{\lambda \in \text{RegLoc}(\Gamma)} \sum_{p \in P^*} \rho(\Gamma, \lambda) \cdot \eta(\Gamma, \lambda, p) \cdot \otimes_{q < p} \Gamma(\lambda, q) \otimes d(\Gamma(\lambda, p)) \otimes \otimes_{q > p} \Gamma(\lambda, q) \\ &= \sum_{\lambda \in \text{RegLoc}(\Gamma)} \sum_{p \in P^*} \sum_{e \in E_{\Gamma(\lambda, p)}^{\text{contr}}} \rho(\Gamma, \lambda) \cdot \eta(\Gamma, \lambda, p) \cdot \epsilon(\Gamma(\lambda, p), e) \cdot \\ &\quad \cdot \otimes_{q < p} \Gamma(\lambda, q) \otimes (\Gamma(\lambda, p)/e) \otimes \otimes_{q > p} \Gamma(\lambda, q) \\ (33) \quad &= \sum_{(e, \lambda) \in \Omega} \rho(\Gamma, \lambda) \cdot \eta(\Gamma, \lambda, \lambda_E(e)) \cdot \epsilon(\Gamma(\lambda, \lambda_E(e)), e) \cdot (\otimes_{p \in P^*} (\Gamma/e)(\lambda/e, p)). \end{aligned}$$

On the other hand

$$\begin{aligned}
 \Psi(d(\Gamma)) &= \Psi \left(\sum_{e \in E_\Gamma^{\text{contr}}} \epsilon(\Gamma, e) \Gamma / e \right) \\
 &= \sum_{(e, \bar{\lambda}) \in \bar{\Omega}} \epsilon(\Gamma, e) \cdot \rho(\Gamma / e, \bar{\lambda}) \cdot (\otimes_{p \in P^*} (\Gamma / e)(\bar{\lambda}, p)) \\
 (34) \quad &\stackrel{\text{by Lemma 7.12}}{=} \sum_{(e, \lambda) \in \Omega} \epsilon(\Gamma, e) \cdot \rho(\Gamma / e, \lambda / e) \cdot (\otimes_{p \in P^*} (\Gamma / e)(\lambda / e, p))
 \end{aligned}$$

It remains to check that the signs of (33) and (34) agree, which is straightforward. \square

7.4. Associativity of the structure maps. In order to prove that $\hat{\Psi}$ and Ψ can serve to define a cooperad structure, we need to check some associativity condition.

Fix the setting 5.4. Suppose moreover that A is itself linearly ordered, that ν is increasing, and that $P^* \cap A = \emptyset$. Let $\xi: B \rightarrow A$ be an ordered weak partition of a finite set B . Set $B_a := \xi^{-1}(a)$ for $a \in A$. Set also $X^* := \{0\} \otimes X$.

We have then a natural bijection

$$\Pi_{a \in A} B_a \cong \Pi_{p \in P} \Pi_{a \in A_p} B_a.$$

For $p \in P$, the partition ξ restrict to a weak ordered partitions

$$\xi_p: \Pi_{a \in A_p} B_a \rightarrow A_p.$$

Lemma 7.14. *The following diagram is commutative:*

$$\begin{array}{ccc}
 (35) \quad \widehat{\mathcal{D}}(\Pi_{p \in P} (\Pi_{a \in A_p} B_a)) & \xlongequal{\quad} & \widehat{\mathcal{D}}(\Pi_{a \in A} B_a) \\
 \downarrow \hat{\Psi}_{\nu \circ \xi} & & \downarrow \hat{\Psi}_\xi \\
 \widehat{\mathcal{D}}(P) \otimes \otimes_{p \in P} \widehat{\mathcal{D}}(\Pi_{a \in A_p} B_a) & & \widehat{\mathcal{D}}(A) \otimes \otimes_{a \in A} \widehat{\mathcal{D}}(B_a) \\
 \downarrow \text{id} \otimes \otimes_{p \in P} \hat{\Psi}_{\xi_p} & & \downarrow \hat{\Psi}_\nu \otimes \text{id} \\
 \widehat{\mathcal{D}}(P) \otimes \otimes_{p \in P} \left(\widehat{\mathcal{D}}(A_p) \otimes \otimes_{a \in A_p} \widehat{\mathcal{D}}(B_a) \right) & \xrightarrow[\tau]{\cong} & \left(\widehat{\mathcal{D}}(P) \otimes \otimes_{p \in P} \widehat{\mathcal{D}}(A_p) \right) \otimes \otimes_{a \in A} \widehat{\mathcal{D}}(B_a)
 \end{array}$$

where the horizontal isomorphism τ is the obvious reordering of factors (with the usual Koszul sign).

Proof. Straightforward. \square

7.5. Action of the symmetric group on the space of diagrams. Let A be a linearly ordered finite set. This group acts on the set of diagrams on A by, for $\sigma \in \text{Perm}(A)$ and a diagram (A, E, I, s, t) ,

$$\sigma \cdot (A, E, I, s, t) = (A, E, I, \sigma \circ s, \sigma \circ t)$$

where the bijection $\sigma: A \xrightarrow{\cong} A$ is extended to internal vertices by $\sigma(v) = v$ for $v \in I$. The following is immediate:

Proposition 7.15. *There is an induced action of CDGA of $\text{Perm}(A)$ on $\widehat{\mathcal{D}}(A)$ and $\mathcal{D}(A)$.*

7.6. The cooperad structure on $\{\widehat{\mathcal{D}}(n)\}_{n \geq 0}$ and on $\{\mathcal{D}(n)\}_{n \geq 0}$. For an integer $n \geq 0$, set $\widehat{\mathcal{D}}(n) := \widehat{\mathcal{D}}(\underline{n})$ and $\mathcal{D}(n) := \mathcal{D}(\underline{n})$.

Let k and n_1, \dots, n_k be non-negative integers and set $n = n_1 + \dots + n_k$. We have an obvious bijection

$$\underline{n} \cong \coprod_{i \in \underline{k}} \underline{n}_i = \underline{n}_1 \amalg \dots \amalg \underline{n}_k.$$

which corresponds to an obvious ordered partition $\nu: \underline{n} \rightarrow \underline{k}$ which is an increasing function. Therefore we have maps

$$\begin{aligned} \hat{\Psi}: \widehat{\mathcal{D}}(n) &\rightarrow \widehat{\mathcal{D}}(k) \otimes \widehat{\mathcal{D}}(n_1) \otimes \dots \otimes \widehat{\mathcal{D}}(n_k) \quad \text{and} \\ \Psi: \mathcal{D}(n) &\rightarrow \mathcal{D}(k) \otimes \mathcal{D}(n_1) \otimes \dots \otimes \mathcal{D}(n_k). \end{aligned}$$

We also have by Section 7.5 an action of the symmetric groups Σ_n on $\widehat{\mathcal{D}}(n)$ and $\mathcal{D}(n)$.

Define CDGA maps

$$\hat{\eta}: \widehat{\mathcal{D}}(1) \rightarrow \mathbb{K} \quad \text{and} \quad \eta: \mathcal{D}(1) \rightarrow \mathbb{K}$$

by $\hat{\eta}(\mathbf{1}) = 1$ and $\hat{\eta}(\Gamma) = 0$ for a diagram other than the unit, and similarly for η .

Theorem 7.16. *The structure maps $\hat{\Psi}$ and Ψ , the symmetric action, and the counits $\hat{\eta}$ and η described above define:*

- *the structure of a cooperad of graded \mathbb{K} -algebras on $\{\widehat{\mathcal{D}}(n)\}_{n \geq 0}$, and*
- *the structure of a cooperad of CDGAs on $\{\mathcal{D}(n)\}_{n \geq 0}$.*

Proof. The associativity of the structure maps $\hat{\Psi}$ required for a cooperad structure is exactly Lemma 7.14. We have the corresponding associativity for Ψ since, by Proposition 7.4, that structure map is induced by $\hat{\Psi}$. It is easy to check that $\hat{\eta}$ and η are counits. The equivariance is also easy to check. \square

8. THE KONTSEVICH INTEGRAL

The goal of this section is to construct CDGA morphisms

$$I: \mathcal{D}(n) \rightarrow \Omega_{PA}^*(C[n]),$$

called the *Kontsevich integrals*, which will turn out to be quasi-isomorphisms and also morphisms of cooperads “up to homotopy”. In all the section the ground ring is the field of real numbers $\mathbb{K} = \mathbb{R}$.

8.1. Construction of the Kontsevich integral $\widehat{\mathbf{I}}$. Fix a finite set A . We construct a linear map

$$\widehat{\mathbf{I}}: \widehat{\mathcal{D}}(A) \rightarrow \Omega_{PA}(\mathbb{C}[A])$$

as follows.

Let Γ be a diagram on A .

Let $\mathrm{dvol} \in \Omega_{\min}^{N-1}(S^{N-1})$ be the standard normalized volume form on the sphere $S^{N-1} \subset \mathbb{R}^N$ defined as

$$(36) \quad \mathrm{dvol} = \kappa_N \cdot \sum_{i=1}^N (-1)^i t_i dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_N$$

where t_1, \dots, t_N are the standard coordinates in \mathbb{R}^N , $\widehat{dt_i}$ means a missing factor, and $\kappa_N \in \mathbb{R}$ is a normalizing constant such that

$$\int_{S^{N-1}} \mathrm{dvol} = 1.$$

More generally, for any linearly ordered finite set E let

$$(S^{N-1})^E = \prod_{e \in E} S^{N-1},$$

and denote by dvol_E the top volume form in that product,

$$\mathrm{dvol}_E := \times_{e \in E} \mathrm{dvol}_e \in \Omega_{\min}((S^{N-1})^E)$$

where the products are taken in the order of E and dvol_e is the standard normalized volume form on the e -th factor.

For v and w two distinct vertices in V_Γ , recall from (9) the map

$$\theta_{v,w}: \mathbb{C}[V_\Gamma] \rightarrow S^{N-1}.$$

By convention, when $v = w$, we set $\theta_{v,v}$ to be the constant map to a fixed base point of the sphere. For an edge e of Γ we set $\theta_e = \theta_{s_\Gamma(e), t_\Gamma(e)}$ and we define

$$\theta_\Gamma := (\theta_e)_{e \in E_\Gamma}: \mathbb{C}[V_\Gamma] \rightarrow (S^{N-1})^{E_\Gamma}.$$

We have then a minimal form

$$\theta_\Gamma^*(\mathrm{dvol}_{E_\Gamma}) \in \Omega_{\min}(\mathbb{C}[V_\Gamma])$$

which is of degree $|E_\Gamma| \cdot (N-1)$.

By Theorem 5.7 the canonical projection

$$\pi_\Gamma: \mathbb{C}[V_\Gamma] \rightarrow \mathbb{C}[A]$$

is an oriented SA bundle. When $|A| \geq 2$, the fiber of π_Γ is of dimension $N \cdot |I_\Gamma|$ and integration along the fiber gives a map

$$(37) \quad (\pi_\Gamma)_*: \Omega_{\min}^k(\mathbb{C}[V_\Gamma]) \rightarrow \Omega_{PA}^{k-N \cdot |I_\Gamma|}(\mathbb{C}[A]).$$

When $|A| \geq 2$, define

$$(38) \quad \widehat{\mathbf{I}}(\Gamma) := (\pi_\Gamma)_*(\theta_\Gamma^*(\mathrm{dvol}_{E_\Gamma})) \in \Omega_{PA}(\mathbb{C}[A]).$$

If A is empty or a singleton we set

$$(39) \quad \widehat{\mathbf{I}}(\Gamma) := \begin{cases} 1 & \text{if } \Gamma \text{ is the unit diagram,} \\ 0 & \text{otherwise.} \end{cases}$$

The reason to treat separately the case $|A| \leq 1$ is that the dimension of the fiber of π_Γ is smaller than expected when there are internal vertices (see Theorem 5.7.) Therefore we should in those cases consider the pushforward $\pi_{\Gamma*}$ of (37) to be 0. Formula (39) is a clean way to do so.

Lemma 8.1. *For any finite set A , formulas (38) and (39) induce a degree 0 linear map*

$$\widehat{\mathbf{I}}: \widehat{\mathcal{D}}(A) \rightarrow \Omega_{PA}(\mathbb{C}[A]).$$

Proof. For $|A| \leq 1$ it is clear. Suppose that $|A| \geq 2$. It is easy to check that (38) is compatible with the equivalence relation \simeq of Definition 6.5 (and actually it is the compatibility with $\widehat{\mathbf{I}}$ which is the motivation for the definition of \simeq .) We extend it by linearity. It is clearly of degree 0. \square

8.2. $\widehat{\mathbf{I}}$ is a morphism of algebras. In this section we prove:

Proposition 8.2. *$\widehat{\mathbf{I}}$ is a morphism of algebras.*

Proof. If $|A| \leq 1$ then the proposition is obvious. Suppose now that $|A| \geq 2$. Let Γ_1 and Γ_2 be two diagrams on A and suppose that they have disjoint sets of internal vertices and of edges. Consider the pullback

$$(40) \quad \begin{array}{ccc} P & \xrightarrow{q_2} & \mathbb{C}[V_{\Gamma_2}] \\ q_1 \downarrow & \text{pullback} & \downarrow \pi_2 \\ \mathbb{C}[V_{\Gamma_1}] & \xrightarrow{\pi_1} & \mathbb{C}[A]. \end{array}$$

Set $\pi = \pi_i \circ q_i: P \rightarrow \mathbb{C}[A]$, which is an oriented SA bundle as the composite of two oriented SA bundles. We have canonical projections

$$\rho_i: \mathbb{C}[V_{\Gamma_1, \Gamma_2}] \rightarrow \mathbb{C}[V_{\Gamma_i}]$$

for $i = 1, 2$, and, by universal property of the pullback, we get a map

$$\rho: \mathbb{C}[V_{\Gamma_1, \Gamma_2}] \rightarrow P$$

such that $q_i \circ \rho = \rho_i$. The space P can be interpreted as the space of “singular” virtual configurations of points labeled by V_{Γ_1, Γ_2} , where by singular we mean that some points labeled by I_{Γ_1} are allowed to coincide with points labeled by I_{Γ_2} . Then ρ maps a virtual configuration in $\mathbb{C}[V_{\Gamma_1, \Gamma_2}]$ to the corresponding singular configuration in P where the information about the directions between two components labeled by I_{Γ_1} and I_{Γ_2} respectively is lost, and similarly for some information about relative distance of three components. When

$|A| \geq 2$, then ρ is a homeomorphism on its dense image when restricted to the complement of the codimension 1 subspace of $C[V_{\Gamma_1, \Gamma_2}]$ consisting of virtual configurations where some component labeled by I_{Γ_1} is infinitesimally closed to a component labeled by I_{Γ_2} .

The canonical projection

$$\pi': C[V_{\Gamma_1, \Gamma_2}] \rightarrow C[A]$$

is exactly the map $\pi' = \pi \circ \rho$. Since π and π' are oriented SA bundles over $C[A]$, the fibers $\pi^{-1}(x)$ and $\pi'^{-1}(x)$ are compact oriented manifolds, for $x \in C[A]$. It is easy to see that $\rho_*(\llbracket \pi'^{-1}(x) \rrbracket) = \llbracket \pi^{-1}(x) \rrbracket$ because ρ maps the interior of the manifold $\pi'^{-1}(x)$ homeomorphically to a dense subset of $\pi^{-1}(x)$. Therefore, by [12, Proposition 8.9], for any minimal form $\mu \in \Omega_{\min}(P)$, we have

$$(41) \quad \pi_*(\mu) = \pi'_*(\rho^*(\mu)).$$

We have then

$$\begin{aligned} \widehat{\mathbf{I}}(\Gamma_1 \cdot \Gamma_2) &= \pi'_*(\theta_{\Gamma_1 \cdot \Gamma_2}^*(\text{dvol}_{E_{\Gamma_1} \otimes E_{\Gamma_2}})) \\ &= \pi'_*(\rho^*(q_1^* \theta_{\Gamma_1}^*(\text{dvol}_{E_{\Gamma_1}}) \wedge q_2^* \theta_{\Gamma_2}^*(\text{dvol}_{E_{\Gamma_2}}))) \\ &\stackrel{\text{by Equation (41)}}{=} \pi_* \left(q_1^* \theta_{\Gamma_1}^*(\text{dvol}_{E_{\Gamma_1}}) \wedge q_2^* \theta_{\Gamma_2}^*(\text{dvol}_{E_{\Gamma_2}}) \right) \\ &\stackrel{\text{by [12, Proposition 8.13]}}{=} \pi_{1*}(\theta_{\Gamma_1}^*(\text{dvol}_{E_{\Gamma_1}})) \wedge \pi_{2*}(\theta_{\Gamma_2}^*(\text{dvol}_{E_{\Gamma_2}})) \\ &= \widehat{\mathbf{I}}(\Gamma_1) \cdot \widehat{\mathbf{I}}(\Gamma_2). \end{aligned}$$

□

8.3. Vanishing of $\widehat{\mathbf{I}}$ on non-admissible diagrams. In this section we prove

Proposition 8.3. $\widehat{\mathbf{I}}(\mathcal{N}(A)) = 0$.

Corollary 8.4. $\widehat{\mathbf{I}}$ induces a map of algebras

$$\mathbf{I}: \mathcal{D}(A) \rightarrow \Omega_{PA}(C[A]).$$

The proof of Proposition 8.3 consists of the following lemmas.

Lemma 8.5. $\widehat{\mathbf{I}}$ vanishes on diagrams with loops.

Proof. If $|A| \leq 1$ the lemma is obvious. Suppose that $|A| \geq 2$ and let Γ be a diagram with a loop. One of the components of the map θ_Γ to the product $(S^{N-1})^{E_\Gamma}$ is a constant map. Therefore θ_Γ factors through a space of dimension $< (N-1) \cdot |E_\Gamma|$. By [12, Proposition 5.24] we deduce that the pullback of the maximal degree form dvol_{E_Γ} by θ_Γ is zero, and hence the same is true for $\widehat{\mathbf{I}}(\Gamma)$. □

Lemma 8.6. $\widehat{\mathbf{I}}$ vanishes on diagrams with double edges.

Proof. If $|A| \leq 1$ the lemma is obvious. Suppose that $|A| \geq 2$ and let Γ be a diagram with double edges. The two components of the map θ_Γ corresponding to the double edges factor through the diagonal map

$$\Delta: S^{N-1} \rightarrow S^{N-1} \times S^{N-1}.$$

Therefore θ_Γ factors through a space of dimension $< (N-1) \cdot |E_\Gamma|$. We conclude as in the proof of Lemma 8.5. \square

Lemma 8.7. *$\widehat{\mathbf{I}}$ vanishes on diagrams containing some internal vertex not connected to any external vertices.*

Proof. The lemma is trivial if $|A| \leq 1$. Assume that $|A| \geq 2$. Let Γ be a diagram as in the statement. We have a factorization $\Gamma = \Gamma_1 \cdot \Gamma_2$ where Γ_1 is a diagram with at least one internal vertices and such that all edges are between internal vertices. Since $\widehat{\mathbf{I}}$ is a morphism of algebra, it is enough to prove that $\widehat{\mathbf{I}}(\Gamma_1) = 0$. So without loss of generality we assume that $\Gamma = \Gamma_1$.

The canonical projection π_Γ factors as

$$C[V_\Gamma] \xrightarrow{\rho} C[I_\Gamma] \times C[A] \xrightarrow{q} C[A]$$

where ρ is induced by the canonical projections on each factors, and q is the projection on the second factor. Since we have assumed that the edges of Γ are only between internal vertices, there is a factorization $\theta_\Gamma = \theta' \circ \rho$ for some map

$$\theta': C[I_\Gamma] \times C[A] \rightarrow (S^{N-1})^{E_\Gamma}.$$

Since Γ contains at least one internal vertex, Proposition 5.1 implies that

$$\dim(C[I_\Gamma]) \leq N \cdot |I_\Gamma| - N.$$

Therefore for $x \in C[A]$ we have

$$\dim(q^{-1}(x)) < N \cdot |I_\Gamma| = \dim(\pi_\Gamma^{-1}(x))$$

and [12, Proposition 8.12] implies that

$$\widehat{\mathbf{I}}(\Gamma) = \pi_{\Gamma*}(\theta_\Gamma(\mathrm{dvol}_{E_\Gamma})) = \pi_{\Gamma*}(\rho^*(\theta'^*(\mathrm{dvol}_{E_\Gamma}))) = 0.$$

\square

Lemma 8.8. *$\widehat{\mathbf{I}}$ vanishes on diagrams containing some univalent internal vertex.*

Proof. If $|A| \leq 1$ the lemma is trivial. Suppose that $|A| \geq 2$. Let Γ be a diagram with an internal vertex v of valence 1 and let w be the only vertex adjacent to v . Then V_Γ has at last three vertices. Consider the projection

$$\rho: C[V_\Gamma] \rightarrow C[\{v, w\}] \times C[V_\Gamma \setminus \{v\}]$$

induced by the canonical projections on each factor. Since (v, w) is the only edge with endpoint v we have a factorization $\theta_\Gamma = \theta' \circ \rho$ for some map

$$\theta': C[\{v, w\}] \times C[V_\Gamma \setminus \{v\}] \rightarrow (S^{N-1})^{E_\Gamma}.$$

Since v is internal we get a map

$$q: C[\{v, w\}] \times C[V_\Gamma \setminus \{v\}] \rightarrow C[A]$$

obtained as the projection on the second factor followed by the canonical projection, and $\pi_\Gamma = q \circ \rho$. It is easy to see that for $x \in C[A]$,

$$\dim(q^{-1}(x)) < \dim(\pi_\Gamma^{-1}(x)).$$

[12, Proposition 8.12] implies that

$$\widehat{\mathbf{I}}(\Gamma) = \pi_{\Gamma*}(\theta_{\Gamma}(\mathrm{dvol}_{E_{\Gamma}})) = \pi_{\Gamma*}(\rho^*\theta'^*(\mathrm{dvol}_{E_{\Gamma}})) = 0.$$

□

Lemma 8.9. *$\widehat{\mathbf{I}}$ vanishes on diagrams containing some bivalent internal vertex.*

Proof. The lemma is trivial when $|A| \leq 1$.

Assume that $|A| \geq 2$. We will use Kontsevich's trick from [13, Lemma 2.1]. Let Γ be a diagram with an internal vertex i of valence 2 and let v and w be its adjacent vertices. For concreteness, suppose that the two edges at i are oriented as (v, i) and (w, i) , and ordered by $(v, i) < (w, i)$ as the two last edges of the ordered set E_{Γ} .

To give the idea of the proof suppose first that the diagram consists only in these three vertices and two edges. Set $\theta = (\theta_{v,i}, \theta_{w,i})$ which in this special case is exactly θ_{Γ} , and set $\pi = \pi_{\Gamma}$.

Consider the continuous involution

$$\chi: C[\{v, w, i\}] \xrightarrow{\cong} C[\{v, w, i\}]$$

defined on $C(\{v, w, i\})$ by

$$\chi(y) = (y(v), y(w), y(v) + y(w) - y(i))$$

where $y(v) + y(w) - y(i)$ is the point orthogonally symmetric to $y(i)$ with respect to the line passing through $y(v)$ and $y(w)$. This is a semi-algebraic automorphism of degree $(-1)^N$.

Let

$$A: S^{N-1} \rightarrow S^{N-1}$$

be the antipodal map and let

$$\tau: S^{N-1} \times S^{N-1} \rightarrow S^{N-1} \times S^{N-1}$$

be the interchange of factors which is of degree $(-1)^{N-1}$. By construction of χ , the following diagram commutes

$$(42) \quad \begin{array}{ccc} C[\{v, w, i\}] & \xrightarrow{\theta} & S^{N-1} \times S^{N-1} \\ \chi \downarrow & & \downarrow \tau \circ (A \times A) \\ C[\{v, w, i\}] & \xrightarrow{\theta} & S^{N-1} \times S^{N-1} \end{array}$$

By symmetry of dvol , we have $A^*(\mathrm{dvol}) = \pm \mathrm{dvol}$, so

$$(\tau \circ (A \times A))^*(\mathrm{dvol} \times \mathrm{dvol}) = (-1)^{N-1}(\mathrm{dvol} \times \mathrm{dvol})$$

and hence

$$(43) \quad \chi^*\theta^*(\mathrm{dvol}_{E_{\Gamma}}) = (-1)^{N-1}\theta^*(\mathrm{dvol}_{E_{\Gamma}}).$$

On the other hand the restriction of χ to each fiber $\pi^{-1}(x)$, $x \in C[A]$, is a SA-homeomorphism of degree $(-1)^N$. By [12, Proposition 8.9],

$$(44) \quad \pi_*(\chi^*(\theta^*(\mathrm{dvol}_{E_{\Gamma}}))) = (-1)^N \pi_*(\theta^*(\mathrm{dvol}_{E_{\Gamma}})).$$

We deduce that

$$\begin{aligned}
\widehat{\mathbf{I}}(\Gamma) &= \pi_*(\theta^* \mathrm{dvol}_{E_\Gamma}) \\
&\stackrel{\text{by Equation (44)}}{=} (-1)^N \pi_*(\chi^* \theta^* \mathrm{dvol}_{E_\Gamma}) \\
&\stackrel{\text{by Equation (43)}}{=} (-1)^{N-1} (-1)^N \pi_*(\theta^* \mathrm{dvol}_{E_\Gamma}) \\
&= -\widehat{\mathbf{I}}(\Gamma),
\end{aligned}$$

and hence $\widehat{\mathbf{I}}(\Gamma) = 0$.

For the case of a general diagram, consider the fiber product

$$(45) \quad \begin{array}{ccc} P & \longrightarrow & C[\{v, w, i\}] \\ \downarrow & & \downarrow \pi_1 \\ C[V_\Gamma \setminus \{i\}] & \xrightarrow{\pi_2} & C[\{v, w\}] \end{array}$$

where π_1 and π_2 are the canonical projections. Since $\pi_1 \circ \chi = \pi_1$, the automorphism χ of $C[\{v, w, i\}]$ can be mixed with the identity map on $C[V_\Gamma \setminus \{i\}]$ to give an automorphism of P that we also denote by χ . The canonical projections

$$C[V_\Gamma] \rightarrow C[V_\Gamma \setminus \{i\}] \quad \text{and} \quad C[V_\Gamma] \rightarrow C[\{v, w, i\}]$$

induce a map $\rho: C[V_\Gamma] \rightarrow P$. We have a factorization $\pi_\Gamma = \pi \circ \rho$ for some map $\pi: P \rightarrow C[A]$ which is an oriented SA bundle.

Since the only edges with endpoint i are (v, i) and (w, i) , there is a factorization $\theta_\Gamma = \theta \circ \rho$ for some map

$$\theta: P \rightarrow (S^{N-1})^{E_\Gamma \setminus \{(v,i), (w,i)\}} \times S^{N-1} \times S^{N-1}.$$

For each $x \in C[A]$ the restriction of ρ to the interior of $\pi^{-1}(x)$ is an oriented homeomorphism to a dense image in the fiber $\pi^{-1}(x)$. By naturality of integration along the fiber ([12, Proposition 8.9]),

$$(46) \quad \pi_{\Gamma*}(\theta_\Gamma^*(\mathrm{dvol}_{E_\Gamma})) = \pi_*(\theta^*(\mathrm{dvol}_{E_\Gamma})).$$

As for the diagram (42), we have $\theta \circ \chi = (\mathrm{id} \times \tau \circ (A \times A)) \circ \theta$. The rest of the proof is the same as in the special case treated above, starting with Equation (43). \square

Proof of Proposition 8.3. A non admissible diagram satisfies the hypothesis of one of Lemmas 8.5-8.9. \square

8.4. $\widehat{\mathbf{I}}$ and \mathbf{I} are chain maps. All this section is devoted to the proof of the following:

Proposition 8.10. *If $N \neq 2$ then $\widehat{\mathbf{I}}d = d\widehat{\mathbf{I}}$ and $\mathbf{I}d = d\mathbf{I}$*

Let A be a finite set and let Γ be a diagram on A . We will prove that $\widehat{\mathbf{I}}(d(\Gamma)) = d(\widehat{\mathbf{I}}(\Gamma))$, which by Corollary 8.4 implies the result for \mathbf{I} . If $|A| \leq 1$ then it is obvious. Also if Γ is

non-admissible, then by Proposition 8.3 and Lemma 6.17 we have

$$d(\widehat{\mathbf{I}}(\Gamma)) = 0 = \widehat{\mathbf{I}}(d(\Gamma)).$$

So we assume now that $|A| \geq 2$ and Γ is admissible.

From now on we will drop Γ from the notation when it appears as an index: So $\Gamma = (A, I, E, s, t)$, $V := V_\Gamma$, $\pi := \pi_\Gamma$, etc... Also to define easily orientations of some configuration spaces we assume that A is equipped with an arbitrary linear order and that $V = A \otimes I$.

Recall from (21) the fiberwise boundary of π ,

$$\pi^\partial: C^\partial[V] \rightarrow C[A].$$

As $\theta^*(\text{dvol}_E)$ is a cocycle, the definition of $\widehat{\mathbf{I}}(\Gamma)$ at (38) and the fiberwise Stokes formula of [12, Proposition 8.10] imply that

$$(47) \quad d(\widehat{\mathbf{I}}(\Gamma)) = (-1)^{\deg(\Gamma)} \cdot \pi_*^\partial \left((\theta^* \text{dvol}_E) | C^\partial[V] \right).$$

Using the decomposition of the fiberwise boundary of $C[V]$ from Section 5.6, Proposition 5.18 and Proposition 5.17 (2)-(3) yield to

$$(48) \quad \pi_*^\partial = \sum_{W \in \mathcal{W}(V, A)} (\pi_*^\partial | \text{im } \Phi_W)$$

where Φ_W was defined at (20).

Let $W \in \mathcal{W}(V, A)$, that is: $W \subsetneq V$, $|W| \geq 2$, and $A \subset W$ or $|W \cap A| \leq 1$. Consider the projection to the quotient set

$$q: V \rightarrow V/W.$$

The composite

$$(49) \quad (V \setminus W) \cup \{\min(W)\} \hookrightarrow V \xrightarrow{q} V/W$$

is a bijection and we use it to transport the linear order of V to V/W .

We associate to Γ and W two diagrams, Γ' and $\overline{\Gamma}$, as follows. Intuitively, Γ' is obtained by keeping in Γ only the vertices in W and the edges between them, and $\overline{\Gamma}$ is obtained from Γ by contracting all the vertices in W into a single vertex and forgetting the edges between the vertices of W . More precisely, $\Gamma' := (A', I', E', s', t')$ where

- $A' := A \cap W$;
- $I' := I \cap W$;
- $E' := E \cap s^{-1}(W) \cap t^{-1}(W)$;
- $s' = s|_{E'}$ and $t' = t|_{E'}$,

and $\overline{\Gamma} := (\overline{A}, \overline{I}, \overline{E}, \overline{s}, \overline{t})$ with

- $\overline{A} := q(A)$
- $\overline{I} := (V/W) \setminus q(A)$;
- $\overline{E} := E \setminus E'$;
- $\overline{s} = q \circ (s|_{\overline{E}})$ and $\overline{t} = q \circ (t|_{\overline{E}})$.

Set $\bar{\theta} := \theta_{\bar{\Gamma}}$ and $\theta' := \theta_{\Gamma'}$. The following diagram is commutative

$$(50) \quad \begin{array}{ccccc} (S^{N-1})^{\bar{E}} \times (S^{N-1})^{E'} & \xrightarrow{\tau_W \cong} & (S^{N-1})^E \\ \bar{\theta} \times \theta' \uparrow & & \uparrow \theta \\ C[V/W] \times C[W] & \xrightarrow{\Phi_W} & C^\partial[V] \hookrightarrow C[V] \\ & \searrow \pi^\partial \circ \Phi_W & \downarrow \pi^\partial \swarrow \pi \\ & & C[A] \end{array}$$

where τ_W is the obvious reordering of factors which is a homeomorphism since $E = \bar{E} \amalg E'$. The linear orders on V/W and W gives $C[V/W] \times C[W]$ a natural orientation, as well as to the fibers of $\pi^\partial \circ \Phi_W$. Define the sign

$$\text{sign}(\Phi_W) = \pm 1$$

according to whether

$$\Phi_W: C[V/W] \times C[W] \rightarrow C^\partial[V]$$

preserves or reverses orientation. Then Φ_W induces the same change of orientation between the fibers over any $x \in C[A]$.

Define also $\text{sign}(\tau_W) = \pm 1$ by

$$\tau_W^*(\text{dvol}_E) = \text{sign}(\tau_W) \cdot (\text{dvol}_{\bar{E}} \times \text{dvol}_{E'}).$$

The diagram (50) and [12, Proposition 8.9] imply that

$$(51) \quad (\pi^\partial|_{\text{im } \Phi_W})_*(\theta^* \text{dvol}_E) = \text{sign}(\Phi_W) \cdot \text{sign}(\tau_W) \cdot \left((\pi^\partial \circ \Phi_W)_*(\bar{\theta}^*(\text{dvol}_{\bar{E}}) \times \theta'^*(\text{dvol}_{E'})) \right).$$

Set $\bar{\mu} := \bar{\theta}^*(\text{dvol}_{\bar{E}})$ and $\mu' := \theta'^*(\text{dvol}_{E'})$.

If $A \subset W$ we have a canonical projection

$$\pi': C[W] \rightarrow C[A].$$

If $|W \cap A| \leq 1$ then the composite

$$A \hookrightarrow V \xrightarrow{q} V/W$$

is injective and we have an associated canonical projection

$$\bar{\pi}: C[V/W] \rightarrow C[A].$$

Lemma 8.11.

$$(\pi^\partial \circ \Phi_W)_*(\bar{\mu} \times \mu') = \begin{cases} \bar{\pi}_*(\bar{\mu}) \cdot \langle \mu', \llbracket C[W] \rrbracket \rangle & \text{if } |W \cap A| \leq 1; \\ \pm \pi'_*(\mu') \cdot \langle \bar{\mu}, \llbracket C[V/W] \rrbracket \rangle & \text{if } A \subset W. \end{cases}$$

Proof. If $|W \cap A| \leq 1$ then

$$\pi^\partial \circ \Phi_W = \bar{\pi} \circ \text{proj}_1$$

and the desired formula is a consequence of the double pushforward formula of [12, Proposition 8.11].

If $A \subset W$ then

$$\pi^\partial \circ \Phi_W = \pi' \circ \text{proj}_2$$

and the desired formula is again a consequence of [12, Proposition 8.11] with an extra sign because of the interchange of factors. \square

In the following lemmas we compute the expressions

$$\langle \bar{\mu}, \llbracket C[V/W] \rrbracket \rangle \quad \text{and} \quad \langle \mu', \llbracket C[W] \rrbracket \rangle.$$

Lemma 8.12. *If $N \neq 2$ and Γ_0 is a diagram with at least 3 vertices then*

$$(52) \quad \langle \theta_{\Gamma_0}^*(\text{dvol}_{E_{\Gamma_0}}), \llbracket C[V_{\Gamma_0}] \rrbracket \rangle = 0.$$

Proof. In this proof we drop Γ_0 from the notation when it appears as an index, so here $V := V_{\Gamma_0}$, $E := E_{\Gamma_0}$, and $\theta := \theta_{\Gamma_0}$.

We can assume that

$$(53) \quad \deg \theta^*(\text{dvol}_E) = \dim C[V]$$

because otherwise the left hand side of (52) vanishes for degree reasons.

If Γ_0 has an isolated vertex v then θ factors through $C[V \setminus \{v\}]$. Since

$$\dim C[V \setminus \{v\}] < \dim C[V],$$

the left hand side of (52) vanishes for degree reasons.

If Γ_0 as a univalent vertex and $|V| \geq 3$ then the left hand side of (52) vanishes by the same argument as in the second part of the proof of Lemma 8.8 (where the relevant hypothesis is that they are at least three vertices.)

If Γ_0 has a bivalent vertex then the vanishing follows by the same argument as for Lemma 8.9.

Finally, suppose that all the vertices of Γ_0 are at least trivalent, which implies that

$$E \geq \frac{3}{2}|V|.$$

If $N \geq 3$ we get

$$\deg(\theta^*(\text{dvol}_E)) = (N-1) \cdot |E| \geq \frac{3(N-1)}{2}|V| = N \cdot |V| + \frac{N-3}{2} \cdot |V| \geq N \cdot |V| > \dim C[V]$$

which contradicts Equation (53). If $N = 1$ then (53) can neither hold when $|V| \geq 3$. \square

Remark 8.13. It is unclear wheter this lemma is true when $N = 2$. This is the only place in the proof where we need the hypothesis $N \neq 2$.

From now on we assume that $N \neq 2$.

Lemma 8.14. *If $A \subset W$ then*

$$\langle \bar{\mu}, \llbracket C[V/W] \rrbracket \rangle = 0$$

Proof. If $|V/W| \geq 3$ then we apply Lemma 8.12 to $\Gamma_0 = \bar{\Gamma}$.

Otherwise $|V/W| = 2$ and $V = W \cup \{v\}$ for some internal vertex v of Γ . Since Γ is admissible, v is at least trivalent and its adjacent vertices are in W . Therefore $\bar{\Gamma}$ has double edges (even triple) and we conclude as in the proof of Lemma 8.6. \square

Lemma 8.15. *If $|W| \geq 3$ or if W is a pair of non adjacent vertices of Γ then*

$$\langle \mu', \llbracket C[W] \rrbracket \rangle = 0.$$

Proof. If $|W| \geq 3$ apply Lemma 8.12 to $\Gamma_0 = \Gamma'$.

If W is a pair of non adjacent vertices then Γ' has no edges and hence

$$\deg(\mu') = 0 < \dim C[W].$$

\square

Suppose now that W is a pair of adjacent vertices of Γ and that $|W \cap A| \leq 1$. Then the edge e connecting these two vertices is contractible because it has at most one external vertex and it is not a dead end since Γ is admissible. Moreover we have

$$\bar{\Gamma} = \Gamma/e \quad \text{and} \quad \bar{\pi}_*(\bar{\mu}) = \hat{\Gamma}(\Gamma/e)$$

(note that the order of internal vertices in $\bar{\Gamma}$ is the same as for Γ/e because the ordering (49) is compatible with that of $I_{\bar{\Gamma}}$ from Definition 6.9.) Define the sign

$$(54) \quad \eta(e) = \begin{cases} +1 & \text{if } N \text{ is even or } s(e) < t(e) \\ -1 & \text{otherwise.} \end{cases}$$

Also set in that case $\Phi_e := \Phi_W$ and $\tau_e := \tau_W$.

Lemma 8.16. *If W is a pair of vertices connected by a contractible edge e of Γ then*

$$\langle \mu', \llbracket C[W] \rrbracket \rangle = \eta(e).$$

Proof. Γ' consists of a single edge and we have a homeomorphism

$$\theta' = \theta_{s(e), t(e)}: C[\{s(e), t(e)\}] \rightarrow S^{N-1}$$

which is orientation preserving except when N is odd and $t(e) < s(e)$. Thus

$$\langle \mu', \llbracket C[W] \rrbracket \rangle = \eta(e) \cdot \int_{S^{N-1}} \text{dvol} = \eta(e).$$

\square

Collecting (47), (48), (51), Lemma 8.11, and lemmas 8.14-8.16 we get

$$(55) \quad d(\hat{\Gamma}(\Gamma)) = \sum_{e \in E^{\text{contr}}} (-1)^{\deg(\Gamma)} \cdot \text{sign}(\Phi_e) \cdot \text{sign}(\tau_e) \cdot \eta(e) \cdot \hat{\Gamma}(\Gamma/e).$$

On the other hand, by the definition of $d\Gamma$ at (23)

$$(56) \quad \hat{\Gamma}(d(\Gamma)) = \sum_{e \in E^{\text{contr}}} \epsilon(\Gamma, e) \cdot \hat{\Gamma}(\Gamma/e).$$

It remains to compare the signs of (55) and (56). Let e be a contractible edge of Γ .

Lemma 8.17. $\text{sign}(\tau_e) = (-1)^{(N-1) \cdot (\text{pos}(e:E) + |E|)}$.

Proof. If e is the last edge in the order of E then τ_e is the identity map, and hence $\text{sign}(\tau_e) = +1$ which is the expected value since $\text{pos}(e : E) = |E|$.

When one transposes e with a consecutive edge in the linear order of E then both $\text{sign}(\tau_e)$ and $(-1)^{(N-1) \cdot (\text{pos}(e:E) + |E|)}$ changes by a factor $(-1)^{N-1}$. This proves the formula of the lemma in full generality. \square

Lemma 8.18. $\text{sign}(\Phi_e) = (-1)^{N \cdot (\text{pos}(\max(s(e), t(e)): I) + |I|)}$.

Proof. Suppose first that $t(e)$ is the last and $s(e)$ the second last vertices in the linear order of $X \odot I$. Then it is easy to see that

$$\Phi_e: C[V \setminus \{t(e)\}] \times C[\{s(e), t(e)\}] \rightarrow \partial C[V]$$

is orientation preserving, and hence

$$\text{sign}(\Phi_e) = +1 = (-1)^{N \cdot (|I| + |I|)}$$

as expected.

Consider now permutations of the set of vertices and their induced action on the following diagram

$$\begin{array}{ccc} C[V \setminus \{\max(s(e), t(e))\}] \times C[\{s(e), t(e)\}] & \xrightarrow{\Phi_{\{s(e), t(e)\}}} & \partial C[V] \\ \sigma \times \sigma \downarrow & & \downarrow \sigma \\ C[V \setminus \{\max(\sigma(s(e)), \sigma(t(e)))\}] \times C[\{\sigma(s(e)), \sigma(t(e))\}] & \xrightarrow{\Phi_{\{\sigma(s(e)), \sigma(t(e))\}}} & \partial C[V]. \end{array}$$

Inspecting the changes of signs through this diagram for all possible transpositions σ , it is straightforward to check that the formula is true in full generality. \square

By Lemmas 8.17-8.18 we get that the expressions at (55) and (56) are equal. This finishes the proof of Proposition 8.10.

8.5. $\widehat{\mathbf{I}}$ and \mathbf{I} are almost morphisms of cooperads. We would like to have that $\widehat{\mathbf{I}}$ and \mathbf{I} are morphisms of cooperads. For the same reason as we explained in Section 3 this cannot be true since $\Omega_{PA}(C[\bullet])$ is not a cooperad because Ω_{PA} is not comonoidal. But they are almost morphism of cooperads in the following sense:

Proposition 8.19. (1) *With the setting 5.4, the following diagram is commutative*

$$\begin{array}{ccc}
 \widehat{\mathcal{D}}(A) & \xrightarrow{\widehat{\mathbf{I}}} & \Omega_{PA}(C[A]) \\
 \downarrow \widehat{\Psi} & & \downarrow \Phi^* = \Omega_{PA}(\Phi) \\
 & & \Omega_{PA}(\prod_{p \in P^*} C[A_p]) \\
 & & \uparrow \times \\
 \otimes_{p \in P^*} \widehat{\mathcal{D}}(A_p) & \xrightarrow[\otimes_{p \in P^*} \widehat{\mathbf{I}}]{} & \otimes_{p \in P^*} \Omega_{PA}(C[A_p])
 \end{array}$$

where $\widehat{\Psi}$ and Φ are the (co)operadic structure maps associated to the given ordered weak partition of A .

(2) $\widehat{\mathbf{I}}$ is equivariant with respect to the action of $\text{Perm}(A)$;

(3) $\widehat{\mathbf{I}}$ commutes with the counits $\hat{\eta}: \widehat{\mathcal{D}}(1) \rightarrow \mathbb{R}$ and $\Omega_{PA}(C[1]) \xrightarrow{\cong} \mathbb{R}$.

(1)-(3) are also true when we replace $\widehat{\mathcal{D}}$ by \mathcal{D} , $\widehat{\mathbf{I}}$ by \mathbf{I} , $\widehat{\Psi}$ by Ψ , and $\hat{\eta}$ by η .

The rest of the section is devoted to the proof of that proposition. We first establish a few lemmas.

Assume that $|A| \geq 2$. We use the notation and results of Section 5.5. To simplify notation in the rest of this section we will drop Γ from the notation when it appear as an index, so $I := I_\Gamma$, $\pi := \pi_\Gamma$, $E := E_\Gamma$, etc... Also for $p \in P^*$ we will replace the index $\Gamma(\lambda, p)$ by p , as in $V_p := V_{\Gamma(\lambda, p)}$, $\theta_p := \theta_{\Gamma(\lambda, p)}$, etc...

Consider a localization λ of Γ . In Section 5.5 we have studied the relation between the canonical projection π and the operadic map $\Phi := \Phi_\nu$. Consider the pullback diagram (15) and the diagram (16). From those we build the following commutative diagram

$$\begin{array}{ccccc}
 \prod_{p \in P^*} (S^{N-1})^{E_p} & \xrightarrow{\cong \tau_\lambda} & & & (S^{N-1})^E \\
 \uparrow \prod_{p \in P^*} \theta_p & & & & \uparrow \theta \\
 \prod_{p \in P^*} C[V_p] & \xrightarrow[\cong]{\Phi'_\lambda} G_\lambda & \xrightarrow{\quad} G & \xrightarrow{\Phi'} & C[V] \\
 & \searrow \pi'_\lambda & \downarrow \pi' & \text{pullback} & \downarrow \pi \\
 & & \prod_{p \in P^*} C[A_p] & \xrightarrow{\Phi} & C[A]
 \end{array}$$

$\pi_\lambda = \times_{p \in P^*} \pi_p$

where τ_λ is the obvious interchange of factors.

Lemma 8.20.

$$\tau_\lambda^*(\text{dvol}_E) = \rho(E, \lambda) \cdot (\times_{p \in P^*} \text{dvol}_{E_p})$$

where

$$\rho(E, \lambda) := (-1)^{(N-1) \cdot |R_E|}$$

with

$$R_E = \{(e, f) \in E \times E : e < f \text{ and } \lambda_E(e) > \lambda_E(f)\}.$$

Proof. Interchanging two factors S^{N-1} is a map of degree $(-1)^{N-1}$. The factors of $\prod_{p \in P^*} (S^{N-1})^{E_p}$ are ordered as $\otimes_{p \in P^*} E_p$ and the number of transpositions needed to reorder this set as E is the cardinality of R_E . \square

Recall from Definition 5.13 the notion of a normal localization.

Lemma 8.21. *Let λ be a localization of Γ .*

(i) *If λ is normal then for each $p \in P^*$*

$$\widehat{\mathbf{I}}(\Gamma(\lambda, p)) = \pi_{p*}(\theta_p^*(\mathrm{dvol}_{E_p})).$$

(ii) *If λ is not normal then*

$$\left(\otimes_{p \in P^*} \widehat{\mathbf{I}} \right) (\Gamma(\lambda)) = 0.$$

Proof. Suppose that λ is normal. Then for each $p \in P^*$, either $|A_p| \geq 2$, in which case $\widehat{\mathbf{I}}(\Gamma(\lambda, p))$ is given by (38) as expected, or $I_p = \emptyset$ in which case formulas (39) and (38) agree. If λ is not normal then for some $p \in P^*$ we have $|A_p| \leq 1$ and $|I_p| \neq \emptyset$, in which case $\widehat{\mathbf{I}}(\Gamma(\lambda, p)) = 0$ by (39). \square

Define the set

$$\mathrm{NormLoc}(\Gamma) := \{\lambda \in \mathrm{Loc}(\Gamma) : \lambda \text{ is normal}\}.$$

Proof of Proposition 8.19. If $|A| \leq 1$ the proposition is obvious. Suppose that $|A| \geq 2$. The statements about the equivariance and the counits are easy. We prove the commutativity of the diagram of the proposition. Let Γ be a diagram in $\widehat{\mathcal{D}}(A)$. By inspection of the diagram

(57) we get

$$\begin{aligned}
 \Phi^*(\widehat{\mathbf{I}}(\Gamma)) &= \Phi^*(\pi_*(\theta^*(\mathrm{dvol}_E))) \\
 &\stackrel{\text{diagram (15)}}{=} \pi'_*(\Phi'^*(\theta^* \mathrm{dvol}_E)) \\
 &\stackrel{\text{Lemmas 5.11, 5.14, and 5.15}}{=} \sum_{\lambda \in \mathrm{NormLoc}(\Gamma)} \pi'_{\lambda*}((\Phi'|G_\lambda)^*(\theta^*(\mathrm{dvol}_E))) \\
 &\stackrel{\text{Lemma 5.16 and [12, Proposition 8.9]}}{=} \sum_{\lambda \in \mathrm{NormLoc}(\Gamma)} \rho(I, \lambda) \cdot \pi_{\lambda*}(\Phi'^* \Phi'^* \theta^*(\mathrm{dvol}_E)) \\
 &\stackrel{\text{diagram (57)}}{=} \sum_{\lambda \in \mathrm{NormLoc}(\Gamma)} \rho(I, \lambda) \cdot \pi_{\lambda*}((\times_{p \in P^*} \theta_p)^*(\tau_\lambda^*(\mathrm{dvol}_E))) \\
 &\stackrel{\text{Lemma 8.20}}{=} \sum_{\lambda \in \mathrm{NormLoc}(\Gamma)} \rho(I, \lambda) \cdot \rho(E, \lambda) \cdot \pi_{\lambda*}(\times_{p \in P^*} \theta_p)^*(\mathrm{dvol}_{E_p}) \\
 &\stackrel{\text{definition of } \rho(\Gamma, \lambda) \text{ at (26) and Lemma 8.21 (i)}}{=} \sum_{\lambda \in \mathrm{NormLoc}(\Gamma)} \rho(\Gamma, \lambda) \cdot \times_{p \in P^*} (\widehat{\mathbf{I}}(\Gamma(\lambda, p))) \\
 &\stackrel{\text{Lemma 8.21(ii) and definition of } \Gamma(\lambda) \text{ at (27)}}{=} \sum_{\lambda \in \mathrm{Loc}(\Gamma)} (\times_{p \in P^*} \widehat{\mathbf{I}})(\Gamma(\lambda)) \\
 &\stackrel{\text{definition of } \widehat{\Psi} \text{ at (28)}}{=} (\times_{p \in P^*} \widehat{\mathbf{I}})(\widehat{\Psi}(\Gamma)),
 \end{aligned}$$

which proves the commutativity of the diagram.

Since \mathbf{I} is the map induced by $\widehat{\mathbf{I}}$ on the quotient $\mathcal{D}(A) = \widehat{\mathcal{D}}(A)/\mathcal{N}(A)$, the proposition is also true for \mathbf{I} . \square

9. EQUIVALENCE OF THE COOPERADS $\mathcal{D}(\bullet)$ AND $\mathbf{H}^*(\mathbf{C}[\bullet])$

We show in this section that the cooperad $\mathcal{D}(\bullet)$ is weakly equivalent to $\mathbf{H}^*(\mathbf{C}[\bullet]; \mathbb{K})$.

Fix a finite set A . We recall first the computation of the algebra $\mathbf{H}^*(\mathbf{C}[A]; \mathbb{K})$ due to Fred Cohen, [6]. Denote by $[\mathrm{dvol}] \in \mathbf{H}^{N-1}(S^{N-1}; \mathbb{K})$ the cohomological orientation class of the sphere. Set for a, b distinct in A

$$g_{ab} := \theta_{ab}^*([\mathrm{dvol}]) \in \mathbf{H}^{N-1}(\mathbf{C}[A]; \mathbb{K}).$$

Then as graded algebras we have

$$\mathbf{H}^*(\mathbf{C}[A]; \mathbb{K}) = \frac{\wedge(\{g_{ab} : a, b \in A, a \neq b\})}{(3\text{-term relations}; g_{ab}^2; g_{ab} - (-1)^N g_{ba})}$$

where $\wedge(\{g_{ab}\})$ is the free graded commutative \mathbb{K} -algebra generated by the g_{ab} 's, and the *3-term relations* are

$$g_{ab}g_{bc} + g_{bc}g_{ca} + g_{ca}g_{ab}$$

for all distinct $a, b, c \in A$.

For a, b distinct in A denote by

$$\Gamma\langle a, b \rangle$$

the diagram on A with no internal vertices and whose only edge is a chord from a to b . It is an admissible cocycle of degree $N - 1$.

Theorem 9.1. *There exists a quasi-isomorphism of CDGA*

$$\bar{\mathbf{I}}: \mathcal{D}(A) \xrightarrow{\sim} (\mathbf{H}^*(\mathbf{C}[A]; \mathbb{K}), 0)$$

characterized by

$$\bar{\mathbf{I}}(\Gamma\langle a, b \rangle) = g_{ab},$$

and

$$\bar{\mathbf{I}}(\Gamma) = 0$$

when Γ has an internal vertex.

Moreover $\bar{\mathbf{I}}$ is a weak equivalence of cooperads.

The rest of the section is devoted to the proof of that theorem.

Consider the submodule $\mathcal{D}^{(0)}(A)$ of $\mathcal{D}(A)$ generated by admissible diagrams without internal vertices. Then

$$\mathcal{D}^{(0)}(A) = \frac{\wedge(\{\Gamma\langle a, b \rangle : a, b \in A, a \neq b\})}{((\Gamma\langle a, b \rangle)^2; \Gamma\langle a, b \rangle - (-1)^N \Gamma\langle b, a \rangle)}.$$

Therefore we have a surjective algebra map

$$\bar{\mathbf{I}}_0: \mathcal{D}^{(0)}(A) \rightarrow \mathbf{H}^*(\mathbf{C}[A]; \mathbb{K})$$

defined by $\bar{\mathbf{I}}_0(\Gamma\langle a, b \rangle) = g_{ab}$.

Lemma 9.2.

$$\bar{\mathbf{I}}_0(\mathcal{D}^{(0)}(A) \cap d(\mathcal{D}(A))) = 0.$$

Proof. Since $\bar{\mathbf{I}}_0$ is a morphism of algebra it is enough to check the vanishing of the indecomposable coboundaries. Those correspond to the image by d of diagrams Γ consisting of one internal vertex v and edges connecting it to various external vertices. Let say that these external vertices are $1, \dots, n$ and that the edges are (i, v) for $1 \leq i \leq n$. Then $\bar{\mathbf{I}}_0(d(\Gamma)) = 0$ because of the following easy consequence of the 3-term relations:

$$g_{12}g_{23} \dots g_{n-1,n} + g_{23} \dots g_{n-1,n}g_{n,1} + \dots + g_{n,1}g_{12} \dots g_{n-2,n-1} = 0,$$

which can be proved for example by working with a standard \mathbb{K} -module basis of $\mathbf{H}^*(\mathbf{C}[A]; \mathbb{K})$. \square

This lemma implies that we can define the CDGA morphism $\bar{\mathbf{I}}$ by

$$\bar{\mathbf{I}}(\Gamma) = \begin{cases} \bar{\mathbf{I}}_0(\Gamma) & \text{if } \Gamma \text{ has no internal vertices} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that it induces a morphism of cooperads.

Since $\bar{\mathbf{I}}$ induces a surjection in homology, in order to prove that it induces a quasi-isomorphism we only need to establish the following

Lemma 9.3. *There is an abstract isomorphism of graded \mathbb{K} -modules*

$$H_*(\mathcal{D}(A)) \cong H_*(C(A); \mathbb{K}).$$

The proof of this lemma will take the rest of this section.

A diagram Γ on A induces a partition of A into its path connected components and we denote this partition by ν_Γ . In other words two external vertices x and y belong to the same element $C \in \nu_\Gamma$ if and only if they are connected by a path of unoriented edges in Γ . For a partition ν of A denote by

$$\mathcal{D}(A)\langle\nu\rangle$$

the submodule of $\mathcal{D}(A)$ generated by admissible diagrams Γ whose partition of connected components is ν . It is clear that $\mathcal{D}(A)\langle\nu\rangle$ is a subcomplex of $\mathcal{D}(A)$. In the particular case of the undiscrete partition $\nu = \{A\}$, we get the subcomplex of *connected admissible diagrams*

$$\mathcal{D}(A) := \mathcal{D}(A)\langle\{A\}\rangle.$$

It is clear that we have an isomorphism of complexes

$$(58) \quad \mathcal{D}(A) \cong \bigoplus_\nu \otimes_{C \in \nu} \mathcal{D}(C)$$

where the sum runs over all partitions ν of the set A .

The Poincaré series of the homology of the configuration space $C(A)$ is well known to be

$$(59) \quad (1+t)(1+2t) \dots (1+(|A|-1)t)$$

with t of degree $N-1$, and in particular the top degree Betti number is

$$(60) \quad \dim H^{(N-1)(|A|-1)}(C(A); \mathbb{K}) = (|A|-1)!$$

In view of the isomorphism (58) and formulas (59) and (60), Lemma 9.3 will be a direct consequence of the following

Lemma 9.4. *For A non empty*

$$\dim H^i(\mathcal{D}(A)) = \begin{cases} (|A|-1)! & \text{if } i = (N-1) \cdot (|A|-1), \\ 0 & \text{otherwise.} \end{cases}$$

Before proving this lemma, we introduce further submodules. Fix an element $a \in A$ and consider the following submodules of $\mathcal{D}(A)$:

- $\mathcal{U}_0(A)$ is the submodule generated by connected admissible diagrams with a of valence 1 and such that the only edge with endpoint a is contractible;
- $\mathcal{U}_1(A)$ is the submodule generated by connected admissible diagrams with a of valence ≥ 2 ;
- $\mathcal{D}'(A)$ is the submodule generated by all connected admissible diagrams that are not in $\mathcal{U}_0(A) \oplus \mathcal{U}_1(A)$.

It is clear that $\mathcal{D}(A) = \mathcal{D}'(A) \oplus \mathcal{U}_0(A) \oplus \mathcal{U}_1(A)$ and that $\mathcal{D}'(A)$ is a subcomplex of $\mathcal{D}(A)$.

Lemma 9.5. *The inclusion*

$$\mathcal{D}'(A) \hookrightarrow \mathcal{D}(A)$$

is a quasi-isomorphism.

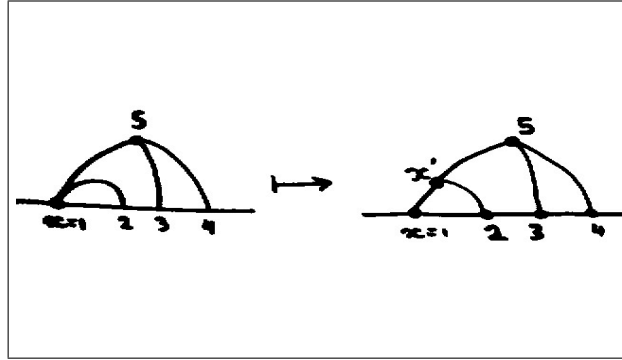


FIGURE 6. Example of blowing vertex $a = 1$ into a contractible edge (a, a') .

Proof. Using the fact that internal vertices in admissible diagrams are at least trivalent, it is easy to check that $\mathcal{U} := \mathcal{U}_0(A) \oplus \mathcal{U}_1(A)$ is also a subcomplex of $\mathcal{D}(A)$. We need to show that \mathcal{U} is acyclic.

Define an increasing filtration on \mathcal{U} where elements of filtration $\leq p$ are the linear combinations of diagrams in \mathcal{U}_0 with less than p edges and diagrams in \mathcal{U}_1 with less than $p - 1$ edges. It is clear that the differential preserves the filtration. Consider the spectral sequence associated to this filtration and which converges to the homology of \mathcal{U} . The differential at the 0th page

$$d^0: \mathcal{U}_0(A) \rightarrow \mathcal{U}_1(A)$$

consists of contracting the only edge with endpoint a . It is an isomorphism because there is an inverse given by “blowing up” the vertex a of a diagram $\Gamma \in \mathcal{U}_1(A)$ into a contractible edge (a, a') as in Figure 6. Therefore the page E^1 of the spectral sequence is trivial and hence \mathcal{U} is acyclic. \square

We are now ready for the

Proof of Lemma 9.4. The proof is by induction on the cardinality of A .

If A is a singleton then $\mathcal{D}'(A) = \mathbb{K} \cdot \mathbf{1}$, where $\mathbf{1}$ is the unit diagram with a single external vertex and no internal vertices, neither edges. Lemma 9.4 is then a consequence of Lemma 9.5.

Let A be of cardinality k and suppose that the lemma has been proved for $< k$ external vertices. Fix $a \in A$. Note that any diagram in $\mathcal{D}'(A)$ has exactly one edge with endpoint a and it is a chord. We have an isomorphism of complexes

$$\mathcal{D}'(A) \cong \oplus_{b \in A \setminus \{a\}} \Gamma\langle a, b \rangle \cdot \mathcal{D}(A \setminus \{a\}).$$

Using Lemma 9.5 we deduce that

$$\dim H^i(\mathcal{D}(A)) = (|A| - 1) \cdot \dim H^{i-N-1}(\mathcal{D}(A \setminus \{a\}))$$

and we conclude using the induction hypothesis. \square

We finish now the

Proof of Lemma 9.3. This is an elementary computation using isomorphism (58), formulas (59) and (60), and Lemma 9.4. \square

This finishes the proof of Theorem 9.1.

10. PROOF OF THE FORMALITY THEOREMS

In this section we prove the theorems of the Introduction. Suppose $N \neq 2$. Let show first that

$$I: \mathcal{D}(A) \rightarrow \Omega_{PA}(C[A])$$

is a weak equivalence. It is a chain map by Proposition 8.10 (here we need $N \neq 2$.) The map induced in cohomology is surjective because, for a, b distinct in A , the one chord diagrams $\Gamma\langle a, b \rangle$ are sent to $\theta_{ab}^*(\text{dvol})$ which correponds clearly to the generators g_{ab} of the cohomology algebra of the configuration space (see Section 9). Since, by formalAD, $H(\mathcal{D}(A)) \cong H^*(C[A])$, we deduce that I is a quasi-isomorphism.

By [12, Proposition 7.1], Ω_{PA} and $A_{PL}(u(-); \mathbb{R})$ are weakly equivalent symmetric monoidal contravariant functors where

$$u: \text{CompactSemiAlg} \rightarrow \text{Top}$$

is the symmetric strongly monoidal forgetful functor. In view of Definition 3.1, all of this combined with Theorem 9.1 and Proposition 8.19 imply that $H(C_N[\bullet]; \mathbb{R})$ is a CDGA model of the operad $C_N[\bullet]$, and hence the same is true for the little N -disks operad. This establishes Theorem 1.2.

We deduce now the stable formality of the operad. Recall from [12, Definition 3.1] the chain complex of semi-algebraic currents

$$C_*: \text{SemiAlg} \rightarrow \text{Ch}_{\mathbb{Z}}.$$

We define the dual of a graded real vector space or of a graded \mathbb{Z} -module, V , as

$$V^{\vee} := \text{Hom}(V, \mathbb{R}),$$

and the dual of a linear map $f: V \rightarrow W$ is denoted by $f^{\vee}: W^{\vee} \rightarrow V^{\vee}$. By [12, Proposition 7.3] we have a natural evaluation map

$$\text{ev}: C_*(X) \otimes \mathbb{R} \rightarrow (\Omega_{PA}(X))^{\vee}$$

which is a weak equivalence when X is a compact semi-algebraic set.

Fix the setting 5.4 and consider the following diagram (in which we write \otimes_{P^*} for $\otimes_{p \in P^*}$)

$$\begin{array}{ccccc}
\otimes_{P^*} C_*(C[A_p]) \otimes \mathbb{R} & \xrightarrow[\simeq]{\otimes_{P^*} \text{ev}} & \otimes_{P^*} (\Omega_{PA}(C[A_p]))^\vee & \xrightarrow[\simeq]{\otimes_{P^*} (I^\vee)} & \otimes_{P^*} (\mathcal{D}(A_p))^\vee \\
\downarrow \times \simeq & & \downarrow \simeq & & \downarrow \simeq \\
& & (\otimes_{P^*} \Omega_{PA}(C[A_p]))^\vee & \xrightarrow[\simeq]{(\otimes_{P^*} I)^\vee} & (\otimes_{P^*} \mathcal{D}(A_p))^\vee \\
& & \uparrow \simeq (\times)^\vee & & \downarrow \Psi_\nu \\
C_*(\prod_{p \in P^*} C[A_p]) \otimes \mathbb{R} & \xrightarrow[\simeq]{\text{ev}} & (\Omega_{PA}(\prod_{p \in P^*} C[A_p]))^\vee & & \\
\downarrow C_*(\Phi_\nu) & & \downarrow (\Omega_{PA}(\Phi_\nu))^\vee & & \\
C_*(C[A]) \otimes \mathbb{R} & \xrightarrow[\simeq]{\text{ev}} & (\Omega_{PA}(C[A]))^\vee & \xrightarrow[\simeq]{I^\vee} & (\mathcal{D}(A))^\vee
\end{array}$$

This diagram is commutative by [12, Proposition 7.3] and Proposition 8.19.

Note that \mathcal{D}^\vee , as the dual of the cooperad of graded differential vector spaces \mathcal{D} , is an operad. The above diagram implies that the operad $C_*(C[\bullet]) \otimes \mathbb{R}$ is weakly equivalent to $(\mathcal{D})^\vee$. By Theorem 9.1, the latter is weakly equivalent to $H_*(C[\bullet]) \otimes \mathbb{R}$. By [12, Proposition 7.2], the symmetric monoidal functors C_* and S_* are weakly equivalent. This proves the stable formality of the little N -disks operad, when $N \neq 2$. For $N = 2$ it has been proved by Tamarkin [21]

We come now to the proof of the relative formality. Let $m \geq 1$ and $N \geq 2m + 1$ be integers with $m \neq 2$. Suppose given a linear isometry

$$\epsilon: \mathbb{R}^m \rightarrow \mathbb{R}^N.$$

Consider the map

$$C_\epsilon := C_\epsilon[A]: C_m[A] \rightarrow C_N[A]$$

that sends a configuration in \mathbb{R}^m to its image by ϵ in \mathbb{R}^N . Clearly it induces a morphism of operads and is equivalent to the morphism induced by ϵ between the little disks operads.

Define the morphism

$$\mathcal{D}_\epsilon: \mathcal{D}_m(A) \rightarrow \mathcal{D}_N(A)$$

by, for a diagram Γ in $\mathcal{D}_m(A)$,

$$\mathcal{D}_\epsilon(\Gamma) = \begin{cases} 1 & \text{if } \Gamma \text{ is the unit diagram} \\ 0 & \text{otherwise.} \end{cases}$$

It induces a morphism of CDGA cooperads. Since $m < N$, $H^*(C_\epsilon; \mathbb{R})$ is the trivial map, *i.e.* it is zero in positive degree and maps the unit of the cohomology algebra to the unit. This combined with Theorem 9.1 implies that the morphism of cooperads \mathcal{D}_ϵ is formal.

We prove now that the following diagram commutes:

$$(61) \quad \begin{array}{ccc} \mathcal{D}_N(A) & \xrightarrow[\simeq]{I_N} & \Omega_{PA}^*(C_N[A]) \\ \mathcal{D}_\epsilon \downarrow & & \downarrow \Omega_{PA}^*(C_\epsilon) \\ \mathcal{D}_m(A) & \xrightarrow[\simeq]{I_m} & \Omega_{PA}^*(C_m[A]). \end{array}$$

Let Γ be an admissible diagram in $\mathcal{D}_N(A)$. If it is the unit diagram then it is clear that

$$(62) \quad \Omega_{PA}^*(C_\epsilon)(I_N(\Gamma)) = I_m(\mathcal{D}_\epsilon(\Gamma)).$$

We assume now that Γ is not a unit. Denote by E its set of edges and by I its set of internal vertices.

Suppose first that moreover each external vertex of Γ is an endpoint of some edge. Since internal vertices are at least trivalent, this implies that

$$|E| \geq \frac{1}{2}(|A| + 3 \cdot |I|).$$

Since $N \geq 2m + 1 \geq 3$, we deduce

$$\begin{aligned} \deg(\Gamma) &= (N-1) \cdot |E| - N \cdot |I| \\ &\geq \frac{N-1}{2}(|A| + 3 \cdot |I|) - N \cdot |I| \\ &= \frac{N-3}{2} \cdot |I| + \frac{N-1}{2} \cdot |A| \\ &\geq |A| \cdot m \\ &> \dim(C_m[A]), \end{aligned}$$

and hence Equation (62) holds for degree reasons.

Consider now a general admissible non unit diagram Γ on A and let $B \subset A$ be the set of external vertices that are the endpoints of some edge of Γ . We have an obvious associated map

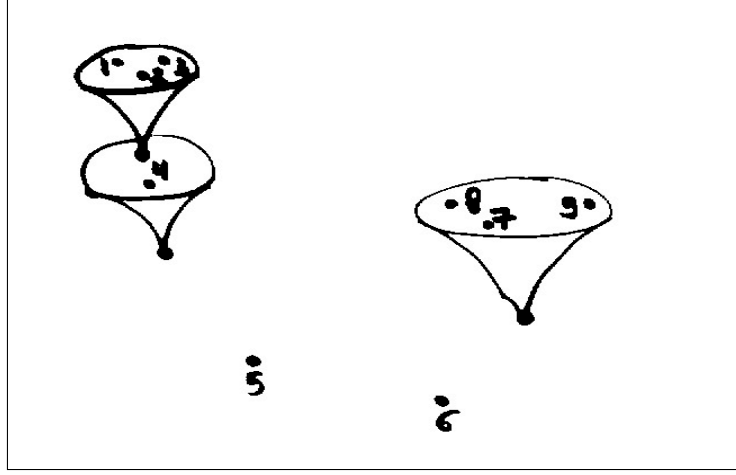
$$\iota: \mathcal{D}_N(B) \rightarrow \mathcal{D}_N(A) \quad , \quad \Gamma' \mapsto \iota(\Gamma')$$

defined by adding to a diagram Γ' in $\mathcal{D}_N(B)$ isolated external vertices labeled by $A \setminus B$. Thus Γ is the image by ι of some diagram $\Gamma' \in \mathcal{D}_N(B)$. The map ι can easily be described in terms of cooperadic operations in an analogous way of the operadic description before Definition 5.6 of the canonical projection

$$\pi: C[A] \rightarrow C[B].$$

It is easy to see that the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}_N(B) & \xrightarrow[\simeq]{I_N} & \Omega_{PA}^*(C_N[B]) \\ \iota \downarrow & & \downarrow \Omega_{PA}^*(\pi) \\ \mathcal{D}_N(A) & \xrightarrow[\simeq]{I_N} & \Omega_{PA}^*(C_N[A]). \end{array}$$

FIGURE 7. A virtual configuration $x_0 \in C[9]$

Since each external vertex of Γ' is the endpoint of an edge, we get by the discussion above that

$$\Omega_{PA}^*(C_\epsilon)I_N(\Gamma') = I_m\mathcal{D}_\epsilon(\Gamma').$$

The commutativity of the last diagram and naturality imply then that Equation (62) holds for $\Gamma = \iota(\Gamma')$. This achieves to prove the commutativity of the diagram (61), which combined with the formality of \mathcal{D}_ϵ implies the formality of the morphism C_ϵ when $m \neq$.

The stable formality of the morphism of operads $C_\epsilon[\bullet]$ is deduced from the unstable formality above exactly as in the absolute case. This establishes Theorem 1.4.

APPENDIX A. PROOF OF THE LOCAL TRIVIALITY OF THE CANONICAL PROJECTIONS

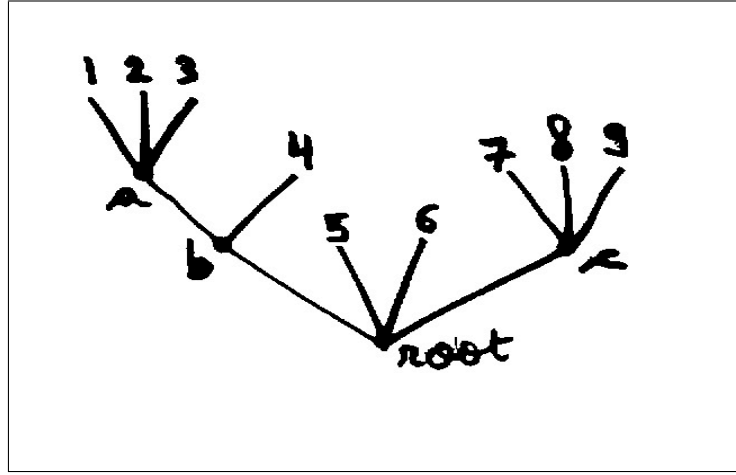
In this section we prove that $\pi: C[V] \rightarrow C[A]$ is a semi-algebraic oriented bundle with fibers of prescribed dimension (see Theorem 5.7.) That the projection $C(V) \rightarrow C(A)$ is a bundle is a classical result due to Fadell and Neurwith [7]. The proof for the compactified version is a little bit more technical because of the existence of a boundary.

The composite of two oriented semi-algebraic bundles is again an oriented semi-algebraic bundle (see [12, Proposition 8.5]), therefore it is enough to prove that

$$\pi: C[n+1] \rightarrow C[n]$$

is an oriented semi-algebraic bundle. For $n \leq 1$ it is trivial, so we assume now that $n \geq 2$.

We give first a rough idea of the proof on an example. Take $n = 9$ and consider the virtual configuration $x_0 \in C[9]$ as in Figure 7 (see [17, Figure 4.2] for an explanation of that figure).

FIGURE 8. The tree T associated to $x_0 \in C[9]$

For this configuration we have proximity relations like the following:

$$x_0(1) \simeq x_0(2) \text{ rel } x_0(4)$$

$$x_0(1) \simeq x_0(4) \text{ rel } x_0(5)$$

etc...

All these proximity relations are encoded in an obvious way by the rooted tree T of Figure 8.

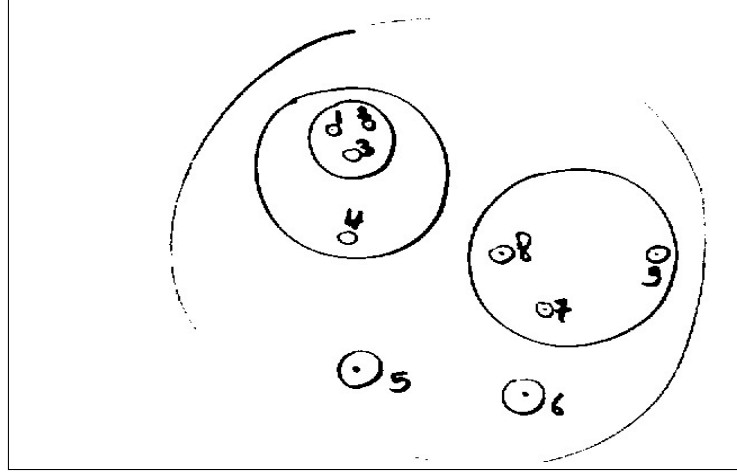
To the virtual configuration x_0 we associate a configuration of nested balls in \mathbb{R}^N as in Figure 9, with one ball B_v for each vertex $v \in \{1, \dots, 9, a, b, c, \text{root}\}$ of the tree T , so that $B_v \subset B_w$ iff w is below v in the tree and such that any two balls are either disjoint or one is contained in the other. The centers of the balls labeled by the leaves define a configuration $x_1 = (x_1(1), \dots, x_1(9)) \in C(9)$. We also assume that each ball is centered at the barycenter of the centers of the balls immediately contained in that one. For example B_b is centered at the barycenter of the centers of B_a and B_4 . Also B_{root} is centered at the origin.

Imagine now a self map

$$\phi_r: \mathbb{R}^N \rightarrow \mathbb{R}^N$$

parametrized by $0 < r \leq 1$ whose effects is iteratively to shrink each ball B_v by a homothety of factor r and extended gently up to the identity map outside of a small neighborhood of the ball. The image of the configuration x_1 by ϕ_r tends to the virtual configuration x_0 as $r \rightarrow 0$.

Consider now a point z anywhere inside the outermost closed ball B_{root} but outside of the innermost open balls B_i for $1 \leq i \leq 9$. Let $y_1 = (x_1, z) \in C(10)$ be the configuration obtained by adjoining the point z to the configuration x_1 . Then the image of y_1 by $\lim_{r \rightarrow 0} \phi_r$ gives an element in the fiber $\pi^{-1}(x_0) \subset C[10]$. By choosing with care the maps ϕ_r we can ensure that all the fiber will be covered by such z 's, giving a homeomorphism $F \cong \pi^{-1}(x_0)$ where F is a closed ball with 9 small disjoint open balls removed.

FIGURE 9. A configuration of nested balls associated to $x_0 \in C[9]$

Allow now the centers of the nested balls to move a bit around their initial value, but preserving the barycentric relations. Moreover bound below the shrinking of each ball B_v by some parameter $\tau(v) \in [0, 1]$, for v a vertex other than the root or a leaf $i = 1, \dots, 9$. Applying then ϕ_r to the configuration of the centers of the balls labeled by the leaves and letting $r \rightarrow 0$, this describes a neighborhood V of x_0 in $C[9]$. A parametrized (by V) version of the above construction will then give a trivialization $V \times F \cong \pi^{-1}(V)$ of π over V . This trivialization can be made semi-algebraic and this will prove that π is a semi-algebraic bundle with an oriented compact generic fiber F .

After giving the general idea on this example, we come now to the details of the proof of Theorem 5.7.

A *rooted tree* T with labels in \underline{n} is a tree (i.e. an isomorphism class of a simply connected 1-dimensional finite simplicial complex) with one distinguished vertex called the *root* of valence ≥ 2 and such that none of the other vertices is bivalent. The univalent vertices are called the *leaves* and are in bijection with the set \underline{n} . An example is given in Figure 8 for $n = 9$.

Denote by \mathcal{V} the set of vertices of the tree T . The leaves are identified with the subset $\underline{n} \subset \mathcal{V}$. Set $\mathcal{V}_0 := \mathcal{V} \setminus \{\text{root}\}$, $\mathcal{V}^* := \mathcal{V} \setminus \underline{n}$ and $\mathcal{V}_0^* := \mathcal{V}^* \cap \mathcal{V}_0$. Define a partial order on \mathcal{V} by letting $w \leq v$ when the shortest path joining v to the root contains w . We write $w < v$ when $w \leq v$ and $w \neq v$. The root is then the minimum of \mathcal{V} . Two vertices v_1, v_2 are *not comparable* if neither $v_1 \leq v_2$, nor $v_2 \leq v_1$. For a non-root vertex v we define its *predecessor*

$$\text{pred}(v) := \max\{w \in \mathcal{V} : w < v\}.$$

For a non-leaf vertex w we define its *output set*

$$\text{output}(w) := \{v \in \mathcal{V} : w = \text{pred}(v)\}.$$

The *height function*

$$\text{height} : \mathcal{V} \rightarrow \mathbb{N}$$

is defined by $\text{height}(\text{root}) = 0$ and $\text{height}(v) = \text{height}(\text{pred}(v)) + 1$ when v is not the root. For example, in Figure 8 we have: $b \leq 1$; $\text{pred}(4) = b$; $\text{output}(\text{root}) = \{b, 5, 6, c\}$; b and 7 are not comparable; and $\text{height}(a) = 2$.

For a rooted tree T with leaves labeled by \underline{n} and set of vertices \mathcal{V} , set

$$C_T := \prod_{w \in \mathcal{V}^*} C(\text{output}(w)).$$

Let $\xi = (\xi^w)_{w \in \mathcal{V}^*} \in C_T$. Thus, for $w \in \mathcal{V}^*$,

$$\xi^w : \text{output}(w) \hookrightarrow \mathbb{R}^N$$

with $\sum_{v \in \text{output}(w)} \xi^w(v) = 0$ and $\sum_{v \in \text{output}(w)} \|\xi^w(v)\| = 1$. For $v \in \mathcal{V}_0$ we set

$$\xi(v) := \xi^{\text{pred}(v)}(v).$$

For $r > 0$ and $v \in \mathcal{V}$, define

$$(63) \quad x(\xi, r, v) := \sum_{\substack{w \in \mathcal{V}_0 \\ w \leq v}} \xi(w) \cdot r^{\text{height}(w)}.$$

Notice that for $r > 0$ small enough,

$$(x(\xi, r, i)_{1 \leq i \leq n})$$

is a configuration in $C(n)$. Define

$$h_T : C_T \rightarrow C[n]$$

by

$$h_T(\xi) = \lim_{r \rightarrow 0^+} (x(\xi, r, i)_{1 \leq i \leq n}).$$

Then h_T is a homeomorphism on its image and the family of $\{\text{im}(h_T)\}$, indexed by all rooted trees T with labels in \underline{n} , gives a stratification of $C[n]$. The maximal stratum is $C(n)$ which is the image of h_{T_0} where T_0 is the tree for which all leaves are of height 1.

Let $x_0 \in C[n]$. Our first goal is to build a neighborhood V of x_0 over which π will be trivial. We have $x_0 = h_T(\xi_0)$ for some tree T and some $\xi_0 \in C_T$.

For a finite set A of at least two elements and for $\xi \in C(A)$ (seen as in (8) in Section 5.1) define

$$\delta(\xi) := \min\{\|\xi(a) - \xi(b)\| : a, b \in A, a \neq b\}$$

which belongs to $(0, 2]$. Set

$$r_1 := \frac{1}{40} \min\{\delta(\xi_0^w) : w \in \mathcal{V}^*\},$$

and set

$$W := \{\xi \in C_T : \forall v \in \mathcal{V}_0, \|\xi(v) - \xi_0(v)\| \leq r_1^{n+1}\}$$

which is a compact neighborhood of ξ_0 in C_T .

Consider now any function

$$\tau : \mathcal{V}_0^* \rightarrow [0, r_1]$$

that we extend to \mathcal{V} by $\tau(\text{root}) = 0$ and $\tau(i) = 0$ for $1 \leq i \leq n$. Define for $\xi \in W$, $0 \leq r \leq r_1$ and $v \in \mathcal{V}$,

$$x(\xi, \tau, r, v) := \sum_{\substack{w \in \mathcal{V}_0 \\ w \leq v}} \left(\xi(w) \cdot \prod_{\substack{u \in \mathcal{V} \\ u \leq w}} \max(r, \tau(u)) \right)$$

Note that $x(\xi, \tau, r, w)$ is the barycenter of the points $x(\xi, \tau, r, v)$ for $v \in \text{output}(w)$.

Finally define

$$\begin{aligned} \Phi: W \times [0, r_1]^{\mathcal{V}_0^*} &\rightarrow C[n] \\ (\xi, \tau) &\mapsto \lim_{r \rightarrow 0+} (x(\xi, \tau, r, i))_{1 \leq i \leq n} \end{aligned}$$

Lemma A.1. *Φ is a semi-algebraic homeomorphism on its image which is a compact neighborhood of x_0 in $C[n]$.*

Proof. We prove the injectivity of Φ . Let y be in the image of Φ , that is

$$y = \lim_{r \rightarrow 0+} (x(\xi, \tau, r, i))_{1 \leq i \leq n}.$$

We want to show that we can uniquely determine ξ and τ from y . Define inductively, for $w \in \mathcal{V}$, $y(w)$ as the (virtual) barycenter of the points $y(v)$ for $v \in \text{output}(w)$. The function τ can be recovered by comparing the radii of the various sets $\{y(v) : v \in \text{output}(w)\}$, $w \in \mathcal{V}^*$. Then since

$$y(v) = \lim_{r \rightarrow 0+} x(\xi, \tau, r, v),$$

we recover easily ξ . This proves the injectivity of Φ .

Since the domain of Φ is compact, Φ is a homomorphism on its image and it is clear that this image is a neighborhood of x_0 . \square

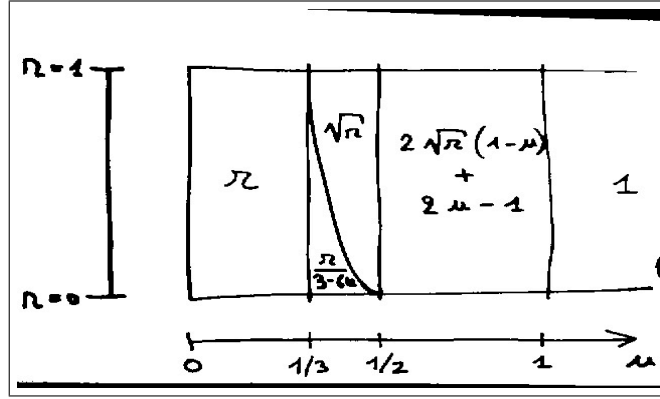
We denote this compact neighborhood of x_0 by

$$V := \Phi(W \times [0, 1]^{\mathcal{V}_0^*}).$$

We will prove the triviality of π over V . In order to do so we will build a configuration of nested balls of centers $x_1(v)$ depending on $\xi \in W$ and of suitable radii $\epsilon(v)$. We will then define some self-maps ϕ_r of \mathbb{R}^N which will shrink these balls and this will lead to the desired trivialization of $\pi^{-1}(V)$. We denote by $B(x, r)$ and $B[x, r]$ the open and the closed ball in \mathbb{R}^N of center x and radius r .

We start by defining a suitable morphism shrinking a given ball. Define first a semi-algebraic function

$$\begin{aligned} g: [0, 1] \times \mathbb{R}_+ &\rightarrow [0, 1] \\ (r, u) &\mapsto g(r, u) \end{aligned}$$

FIGURE 10. Definiton of the function $(r, u) \mapsto g(r, u)$

by

$$g(r, u) = \begin{cases} r & \text{if } 0 \leq u \leq 1/3 \\ \frac{r}{3-6u} & \text{if } 1/3 \leq u \leq 1/2 \text{ and } \sqrt{r} \leq 3-6u \\ \sqrt{r} & \text{if } 1/3 \leq u \leq 1/2 \text{ and } \sqrt{r} \geq 3-6u \\ 2\sqrt{r}(1-u) + 2u - 1 & \text{if } 1/2 \leq u \leq 1 \\ 1 & \text{if } u \geq 1 \end{cases}$$

In other words the function g is determined by the picture of Figure 10 where the curve inside the second rectangle is the parabola $\sqrt{r} = 3 - 6u$.

For $c \in \mathbb{R}^n$, $\epsilon > 0$, and $r \geq 0$ define

$$(64) \quad \begin{aligned} \phi_r^{c,\epsilon}: \mathbb{R}^N &\rightarrow \mathbb{R}^N \\ x &\mapsto \phi_r^{c,\epsilon}(x) = c + (x - c) \cdot g(r, \frac{\|x - c\|}{\epsilon}) \end{aligned}$$

The properties of this map that we will need are summarized in the following:

Lemma A.2. *For $c \in \mathbb{R}^n$, $\epsilon > 0$, and $r \geq 0$, the map $x \mapsto \phi_r^{c,\epsilon}(x)$ has the following properties:*

- (1) *it is radial centered at c ;*
- (2) *it is the identity outside of $B(c, \epsilon)$;*
- (3) *it is semi-algebraic and continuous in the variables c, r, ϵ, x ;*
- (4) *for $r > 0$, it is a homeomorphism;*
- (5) *for $r = 0$, its restriction to $\mathbb{R}^N \setminus B[c, \epsilon/2]$ is a homeomorphism on $\mathbb{R}^N \setminus \{c\}$, and $\phi_0^{c,\epsilon}(B[c, \epsilon/2]) = \{c\}$;*
- (6) *its restriction to $B[c, \epsilon/3]$ is a homothety of rate r ;*
- (7) *if $r > 0$ and $x(1), \dots, x(m)$ are distinct points in $B[c, \epsilon/3]$ then $(\phi_r^{c,\epsilon}(x(1)), \dots, \phi_r^{c,\epsilon}(x(m)))$ defines a configuration in $C(m)$ which does not depend on r ;*

- (8) *with the same hypotheses as in (7), if z_1, z_2 are two distinct points in $B(c, \epsilon/2)$ and are different from the $x(p)$'s for $1 \leq p \leq m$, then*

$$y_i := \lim_{r \rightarrow 0+} (\phi_r^{c, \epsilon}(x(1)), \dots, \phi_r^{c, \epsilon}(x(m)), \phi_r^{c, \epsilon}(z_i))$$

defines two different configurations y_1 and y_2 in $C(m+1)$;

- (9) *with the same hypotheses as in (7), if $z \in \mathbb{R}^N \setminus B(c, \epsilon/2)$ then*

$$y := \lim_{r \rightarrow 0+} (\phi_r^{c, \epsilon}(x(1)), \dots, \phi_r^{c, \epsilon}(x(m)), \phi_r^{c, \epsilon}(z))$$

defines a virtual configuration in $C[m+1]$ such that $y(p) \simeq y(q) \text{ rel } y(m+1)$ for $1 \leq p, q \leq m$;

- (10) *with the same hypotheses as in (7), if z_1, z_2 are two distinct points in \mathbb{R}^n and are different from the $x(p)$'s for $1 \leq p \leq m$, then*

$$y_i := \lim_{r \rightarrow 0+} (\phi_r^{c, \epsilon}(x(1)), \dots, \phi_r^{c, \epsilon}(x(m)), \phi_r^{c, \epsilon}(z_i))$$

defines two different configurations y_1 and y_2 in $C[m+1]$.

Proof. Straightforward using the definition of $\phi_r^{c, \epsilon}$ and g . □

We define now the centers $x_1(v)$ and the radii $\epsilon(v)$ of the balls that we will consider. Suppose given $\xi \in W$ and recall the map x defined at (63). For $v \in \mathcal{V}$, we set

$$(65) \quad x_1(v) := x(\xi, r_1, v)$$

and

$$\epsilon(v) := 4 \cdot r_1^{\text{height}(v)+1}.$$

Lemma A.3. *If $w < v$ in \mathcal{V} then*

$$B[x_1(v), \epsilon(v)] \subset B[x_1(w), \epsilon(w)/3].$$

If v_1 and v_2 are not comparable in \mathcal{V} then

$$B[x_1(v_1), \epsilon(v_1)] \cap B[x_1(v_2), \epsilon(v_2)] = \emptyset.$$

Proof. To simplify the notation, we set $r = r_1$ in this proof. Note that $r \leq 1/20$ because $\delta(\xi_0^w) \leq 2$.

For $w < v$,

$$x_1(v) = x_1(w) + \sum_{w < u \leq v} \xi(u) \cdot r^{\text{height}(u)}.$$

Therefore

$$\begin{aligned}
\|x_1(v) - x_1(w)\| + \epsilon(v) &\leq \sum_{w < u \leq v} \|\xi(u)\| \cdot r^{\text{height}(u)} + 4 \cdot r^{\text{height}(v)+1} \\
&\leq \left(\sup_{w < u \leq v} \|\xi(u)\| \right) \cdot \frac{r^{\text{height}(w)+1}}{1-r} + 4 \cdot r^{\text{height}(v)+1} \\
&\leq r^{\text{height}(w)+1} \left(\frac{1}{1-r} + 4 \cdot r \right) \\
&\leq (4/3) \cdot r^{\text{height}(w)+1} \\
&= \epsilon(w)/3.
\end{aligned}$$

This proves the first part of the lemma.

We come to the second part. Suppose first that v_1 and v_2 have a common predecessor w . Then

$$\begin{aligned}
\|\xi(v_2) - \xi(v_1)\| &\geq \|\xi_0(v_2) - \xi_0(v_1)\| - \|\xi(v_1) - \xi_0(v_1)\| - \|\xi(v_2) - \xi_0(v_2)\| \\
&\geq \delta(\xi_0^w) - 2 \cdot r^{n+1} \\
&\geq 40 \cdot r - 2 \cdot r^{n+1} \\
&\geq 38 \cdot r.
\end{aligned}$$

Since $\text{height}(v_1) = \text{height}(v_2)$ we get

$$\begin{aligned}
\|x_1(v_1) - x_1(v_2)\| &= \|\xi(v_1) - \xi(v_2)\| \cdot r^{\text{height}(v_1)} \\
&\geq 38 \cdot r \cdot r^{\text{height}(v_1)} \\
&> 2 \cdot \epsilon(v_1) \\
&= \epsilon(v_1) + \epsilon(v_2).
\end{aligned}$$

This implies the desired formula when v_1 and v_2 have a common predecessor.

For the general case, notice that, since v_1 and v_2 are not comparable, there exists $w_1 \leq v_1$ and $w_2 \leq v_2$ such that w_1 and w_2 have a common predecessor. Combining the argument above with the fact that, by the first part of the proposition, $B[x_1(v_i), \epsilon(v_i)] \subset B[x_1(w_i), \epsilon(w_i)]$, for $i = 1, 2$, we deduce the desired formula. \square

Fix $\xi \in W$ and $\tau \in [0, r_1]^{\mathcal{V}^*}$. Recall that we extend τ to \mathcal{V} by 0 on the root and the leaves. For $v \in \mathcal{V}$ and $0 \leq r \leq r_1$, set

$$\phi_r^v := \phi_{\max(r, \tau(v))}^{x_1(v), \epsilon(v)}.$$

Note that ϕ_r^v depends on ξ and τ even if does not appear in the notation. Then ϕ_r^v is a self-map of \mathbb{R}^N which is the identity outside of $B[x_1(v), \epsilon(v)]$, and shrinks the ball $B[x_1(v), \epsilon(v)/3]$ by a homothety of rate $\max(r, \tau(v))$. We will compose all these maps ϕ_r^v for $v \in \mathcal{V}$.

If $v_1, v_2 \in \mathcal{V}$ are two distinct vertices of the same height, then they are non comparable and the second part of Lemma A.3 implies that

$$(66) \quad \phi_r^{v_1} \circ \phi_r^{v_2} = \phi_r^{v_2} \circ \phi_r^{v_1}.$$

Let $h_{\max} := \max\{\text{height}(v) : v \in \mathcal{V}\}$. For $0 \leq h \leq h_{\max}$ we define

$$\phi_r^{[h]} := \circ_{\substack{v \in \mathcal{V}_0 \\ \text{height}(v)=h}} \phi_r^v$$

which is the composite of the maps ϕ_r^v for all vertices v of height h , the order of composition being irrelevant because of (66). Finally we set

$$\phi_r := \phi_r^{[0]} \circ \phi_r^{[1]} \circ \dots \circ \phi_r^{[h_{\max}]}$$

which is a selfmap of \mathbb{R}^N whose effect is to shrink iteratively the various balls of center $x_1(v)$.

Set

$$F := B[0, n+1] \setminus \cup_{i=1}^n B(i, 1/4)$$

which will serve as a generic fiber of π . For $\xi \in W$, set also

$$F_\xi := B[x_1(\text{root}), \epsilon(\text{root})/2] \setminus \cup_{i=1}^n B(x_1(i), \epsilon(i)/2).$$

It is easy to build semi-algebraic homeomorphisms

$$\Theta_\xi : F \xrightarrow{\cong} F_\xi$$

that depends continuously and semi-algebraically on ξ .

Define

$$\hat{\Phi} : W \times [0, r_1]^{\mathcal{V}_0^*} \times F \rightarrow C[n+1]$$

by

$$\hat{\Phi}(\xi, \tau, z_0) := \lim_{r \rightarrow 0+} (\phi_r(x_1(1)), \dots, \phi_r(x_1(n)), \phi_r(\Theta_\xi(z_0))).$$

We have $\phi_r(x_1(i)) = x(\xi, \tau, r, i)$, and hence the following diagram commutes

$$\begin{array}{ccc} W \times [0, r_1]^{\mathcal{V}_0^*} \times F & \xrightarrow{\hat{\Phi}} & C[n+1] \\ \text{proj} \downarrow & & \downarrow \pi \\ W \times [0, r_1]^{\mathcal{V}_0^*} & \xrightarrow{\Phi} & C[n]. \end{array}$$

We want to show that $\hat{\Phi}$ is a homeomorphism on $\pi^{-1}(V)$. Fix $(\xi, \tau) \in W \times [0, r_1]^{\mathcal{V}_0^*}$. It is enough to show that we have a homeomorphism

$$\begin{aligned} \hat{\phi} : F_\xi &\xrightarrow{\cong} \pi^{-1}(\Phi(\xi, \tau)) \\ z &\mapsto \hat{\Phi}(\xi, \tau, \Theta_\xi^{-1}(z)) \end{aligned}$$

We show first that $\hat{\phi}$ is injective. Let z_1, z_2 be two distinct elements in F_ξ . Set $y_i = \hat{\phi}(z_i) \in C[n+1]$ for $i = 1, 2$. We treat different cases.

- Suppose that there exists a vertex $v \in \mathcal{V}$ such that $z_1 \in B(x_1(v), \epsilon(v)/2)$ but $z_2 \notin B(x_1(v), \epsilon(v)/2)$ (or the symmetric). By definition of F_ξ , v is not a leaf. Thus v has at least two distinct outputs and we choose two leaves p and q above each of these outputs. Using Lemma A.2 (8)-(9), we get

$$y_1(p) \not\sim y_1(q) \text{ rel } y_1(n+1)$$

but

$$y_2(p) \simeq y_2(q) \text{ rel } y_2(n+1).$$

Thus $y_1 \neq y_2$.

- Suppose that the highest vertex $v \in \mathcal{V}$ such that $z_1 \in B(x_1(v), \epsilon(v)/2)$ is the same for which z_2 has this property. Choose again two leaves p, q above two distinct outputs of v . Set

$$\phi_r^{\geq v} := \phi_r^{[\text{height}(v)]} \circ \phi_r^{[\text{height}(v)+1]} \circ \dots \circ \phi_r^{[\text{height}_{\max}]}$$

By Lemma A.2 (8) one sees that

$$\lim_{r \rightarrow 0} (\phi_r^{\geq v}(z_i), \phi_r^{\geq v}(x_1(p)), \phi_r^{\geq v}(x_1(q)))$$

defines two distinct configurations in $C(3)$. Applying then

$$\lim_{r \rightarrow 0} \phi_r^{[0]} \circ \dots \circ \phi_r^{[\text{height}(v)-1]}$$

gives still two distinct configurations in $C(3)$. Therefore the images of y_1 and y_2 by some canonical projection $\pi: C[n+1] \rightarrow C[3]$ are distinct. Thus $y_1 \neq y_2$.

- It remains to treat the case when there is no $v \in \mathcal{V}$ such that $z_i \in B(x_1(v), \epsilon(v)/2)$ neither for $i = 1$, nor for $i = 2$. Then $z_1, z_2 \in \partial B[x_1(\text{root}), \epsilon(\text{root})/2]$ with $x_1(\text{root}) = 0$. In that case

$$\theta_{1,n+1}(y_i) = z_i / \|z_i\|$$

and these two values are distinct. Thus $y_1 \neq y_2$.

This proves that $\hat{\phi}$ is injective. We come to the proof of the surjectivity. Since F_ξ and $\pi^{-1}(\Phi(\xi, \tau))$ are compact connected manifolds with a non-empty boundary, it is enough to show that $\hat{\phi}$ is surjective on the boundary of the fiber. This boundary consists of virtual configurations $y \in C[n+1]$ such that:

- (a) either for some $1 \leq i \leq n$ and for all $j \in \underline{n} \setminus \{i\}$, we have: $y(i) \simeq y(n+1) \text{ rel } y(j)$;
- (b) or for all $1 \leq i, j \leq n$, we have: $y(i) \simeq y(j) \text{ rel } y(n+1)$.

It is clear that $\hat{\phi}$ maps $\partial B[x_1(i), \epsilon(i)/2]$ surjectively on the boundaries of type (a) and $\partial B[x_1(\text{root}), \epsilon(\text{root})/2]$ on that of type (b).

This achieves to prove that $\hat{\phi}$ is a homeomorphism and hence that that for $n \geq 2$, $\pi: C[n+1] \rightarrow C[n]$ is a semi-algebraic bundle. Its generic fiber is F which is a compact oriented manifold of dimension N .

The general case of Theorem 5.7 is obtained by applying iteratively [12, Proposition 8.5].

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